The Cost and Duration of Cash-Balance Pension Plans

David T. Brown
University of Florida and NISA Investment Advisors, L.L.C.
Philip H. Dybvig
Washington University in Saint Louis
William J. Marshall
NISA Investment Advisors, L.L.C.

June, 2000
Working Draft: Comments welcome!

The authors are grateful for helpful suggestions from Chris Appler, Dave Eichhorn, and Greg Trotter.

Abstract

Controversy about the fairness of early transitions from traditional defined-benefit plans to cash-balance plans may have over-shadowed the subtleties of funding a cash-balance pension liability. Because crediting rates of cash-balance liabilities float with market rates, the same techniques used to value and hedge floating-rate bonds provide the present-value cost and effective duration of a cash-balance liability. The present-value cost of funding a liability varies dramatically across the menu of IRS-sanctioned crediting alternatives; the present value per $1.00 of cash balance of funding a liability paying off 30 years from now varies between $0.90 and $1.48, given the yield curve from November 15, 1999. The effective duration of a cash-balance liability also varies dramatically across different crediting rates; the effective duration is typically positive but much shorter than the expected time until retirement or other payment and can vary by a factor of five or so depending on the choice of crediting rate. These results are useful for comparing the costs of different plans, for comparing how different groups are treated in a plan conversion, or for evaluating the riskiness of any mismatch between assets and liabilities for different funding alternatives.

Introduction

Cash-balance pension plans have become very popular since first introduced by Bank of America in 1984. Cash-balance plans are so-called hybrid plans that are regulated as defined-benefit plans (because benefits are based on a formula and not on actual investment results) but resemble defined-contribution plans. In a cash-balance plan, the cash balance is the current lump-sum accrued pension benefit. (Payment of the cash balance on leaving the firm, when the funds might be rolled into an IRA or another employer's plan, is subject to a vesting requirement for some firms, but usually over a relatively short period, no more than five years, as is typical in defined-contribution plans.) An employee's cash balance comes from periodic "pay-related credits" linked to salary and wages, at a rate which usually depends on age and seniority, and "interest-related credits" at an annually-adjusted crediting rate linked to a market rate (usually a constant-maturity bond yield or discount) or the CPI. This sort of plan is easy for employees to understand because the accumulation of cash balance works the same as the accumulation of the balance in a savings account at a bank or a thrift.

The consulting firm PwC Kwasha HR Solutions estimates that between 400 and 1,000 employers have some sort of cash-balance plan in place (see Anand [1999]). A short list of employers that have adopted or intend to adopt cash-balance plans includes Ameritech, AT&T, Avon Products, Bell Atlantic, Bell South, Carolina Power & Light, California State Teachers Retirement System, Citigroup, Chemical Bank, First Chicago NBD, Georgia-Pacific, IBM, Niagara Mohawk, Public Service Electric and Gas, SBC Communications, and the World Bank.

In general, cash-balance plans are user-friendly because employees who switch jobs can take benefits with them when they leave, and because it is easy to communicate what benefits have accrued. However, many early transitions to cash balance plans have generated public controversy. The controversy usually centers around whether workers in different age groups have been treated fairly in the transition to the cash-balance plan, and not in the merits of the new plan.

Whether or not existing transitions are fair, there remains an interesting question addressed here of computing the cost and effective duration of cash-balance liabilities. Recognizing that a cash-balance plan liability is, in effect, a sort of floating-rate bond, modern term-structure theory can be used to value the liability and compute its effective duration. Our unit of analysis is an ideal unit of cash-balance liability to be withdrawn (presumably due to retirement or departure from the plan) at a known maturity date in the future. (Actuarial assumptions could be used to combine these idealized units to mimic the payments of a whole plan.) The cost of funding this unit of liability depends on the current slope of the term structure of interest rates, any margin associated with various IRS-sanctioned crediting assets, time to maturity, and, to a much lesser extent, on the volatility of interest rates. While the mismatch between cost and cash balance is most pronounced when yield curves are very steep or inverted, there is nonetheless a significant discrepancy even for a yield curve that is relatively flat with a hump.

For example, given a "benchmark" yield curve from November 15, 1999, the market-value cost of a cash-balance liability, maturing in twenty years and crediting at the thirty-year par Treasury yield, is \$0.92 per dollar of cash balance, which amounts to -41 basis points per year. The market-value cost of funding this liability if it credits at the ten-year yield is about \$0.97 per dollar of cash balance (-15 bp/year) and the market value cost of funding this liability if it credits at the one-year rate plus the IRS-guideline margin is \$1.18 per dollar of cash balance (83 bp/year). Since the IRS guidelines permit crediting at the ten-year rate without any margin, it costs the plan sponsor about 100 basis points per year (= 83 - (-15)) to credit at the one-year rate plus the IRS margin rather than at the ten-year rate.

The duration of a cash-balance liability depends primarily on the maturity of the liability and the mismatch between the crediting reset interval and the duration of the crediting asset. For annual resets using the one-year Treasury rate, the duration should be close to zero, that is, the claim's value today is almost insensitive to interest rate movements. This is because increases in the interest rates used to discount the future payments are offset one-for-one by increases in the projected interest crediting rates determining the liability. In the benchmark case with a twenty-year liability credited at the one-year Treasury rate, the computed duration is 0.428 years, which is not exactly zero only because the computations assume quarterly compounding of interest-related credits (while annual compounding would make the computed duration exactly zero).

By contrast, a cash-balance liability with a lump-sum payment in twenty years, credited at the thirty-year par Treasury yield, has an expected duration of about 4 1/2 years. This is significantly less than twenty years (which is the duration of a fixed claim such as a Treasury STRIP maturing in twenty years), because there is significant cancellation of movements in rates used for crediting and discounting. However, the cancellation is incomplete because the thirty-year Treasury yield moves less than one-for-one with changes in the appropriate discount rate. In other words, increases in the interest rates used to discount the future payments are offset only partially by increases in the projected crediting rates determining the liability.

The extent to which the increase in future crediting rates offsets the increase in the discount factor depends critically on the time to maturity of the crediting asset and the maturity of the liability. Understanding how this works is also related to our historical knowledge about how interest rates are related. For example, the increase in the projected future ten-year bond yields over the next ten years following an upward shift in the term structure of interest rates is greater than the expected increase in the projected future thirty-year bond yields over the same ten years. This is why, given the benchmark yield curve, the effective duration of a liability accruing at the ten-year rate is only about 2 1/3 years, while the effective duration of a liability accruing at the thirty-year rate is about 4 1/2 years.

It is useful to mention that the analysis here does not try to include the impact of taxes or liquidity on a firm's valuation of different plans. In other words, valuation treats the pension liabilities as fully liquid cash flows to be funded by liquid assets and computes the value to a tax-exempt investor. In fact, pension liabilities are illiquid and it should be possible to fund them more cheaply with illiquid investments. Unfortunately, the current state of asset pricing does not tell us how to quantify the liquidity discount. (This is becoming a more and more interesting issue given that even Treasury bonds are becoming less liquid, due in part to the government's buy-back program.) The valuation to the firm is further affected by its changing tax status over time and the extent to which the firm is capital-constrained (and faces a liquidity premium internally). Again, the current state of knowledge does not give us a good way to quantify these effects.

The remainder of the paper defines the cost of a cash balance liability and provides an example that develops the intuition for the main results of the paper (Section 1), computes market value costs and effective durations of various liabilities given the benchmark yield curve (Section 2), and presents a general framework for performing precise calculations of cost (Section 3) and effective duration (Section 4).

1 Definitions and an Example

The unit of analysis is the market-value cost (i.e., the appropriate present value at market rates of interest) of a liability at a fixed maturity represented by $1.00 of cash balance today. This cost is the appropriate present value of the final payoff, projecting only future interest-related credits and no pay-related credits. This cost is the fair-market-value analog of the projected benefit obligation (PBO) in traditional defined-benefit plans. In a traditional plan, the PBO, an accounting measure of the value of the benefits that have accrued so far, projects the lump-sum pension benefit based on current years of service and a projected salary at retirement, and discounts this value at a corporate bond rate. (The PBO calculation for traditional defined-benefit plans differs from the market-value cost because the corporate bond rate is probably too high a discount rate for a liability collateralized by the assets in a plan. However, the effective durations and most comparisons of costs across different crediting options are not affected much by this difference. The usual PBO calculation for a cash balance plan is more convoluted since the final obligation does not depend on final salary except indirectly through future pay-related credits.) The specific calculations in this paper assume that it is known for certain when the employee will retire or leave the firm, but these calculations could be merged with an actuarial analysis of departure rates to correspond more closely with traditional PBO calculations (as in Kopp and Sher [1998]).

The analysis in this paper varies time-to-retirement, the market rate underlying the crediting rate, and the margin used in computing the crediting rate. (The analysis does not vary the timing of resets from its customary value of once a year or the details of compounding within a year, both of which should

Table 1: IRS Suggested Guidelines for Crediting Rates

quantity	**standard index**	**associated margin**
discount rate	3-month T-Bills	175 basis points
discount rate	6- or 12-month T-Bills	150 basis points
yield	1-year Treasuries	100 basis points
yield	2- or 3-year Treasuries	50 basis points
yield	5- or 7-year Treasuries	25 basis points
yield	10-year or longer Treasuries	0 basis points
rate of change	CPI-U (urban cost-of-living)	300 basis points

be of minor importance.) The crediting rate is a prespecified market rate, a prespecified market rate plus a margin, or CPI plus a margin. (In principle, there is no reason why there could not be a more complicated nonlinear rule for the crediting rate or a rule based on the average of several rates. However, IRS safe-harbor provisions described below are likely to induce firms to choose among the simple rules analyzed here.) For example, the crediting rate for the California State Teachers Retirement System plan is the average yield on 30-year U.S. Treasury notes during the preceding year.

Table 1 shows the IRS suggestions for margins to be used with different crediting rates (from IRS Notice 96-8, *Internal Revenue Bulletin* **1996-6**, 23–26). A company following these suggestions has a "safe harbor" from the default rules for minimum lump-sum contributions; without the safe harbor the company bears risk through the "whip-saw" effect. ("Whip-saw" is wild fluctuation of the benefit to be paid to workers who leave the plan. See the discussion in Coleman [1998].) The guidelines allow a company to credit at the 3-month Treasury discount rate plus 175 basis points, at the 1-year Treasury yield plus 100 basis points, or at the 30-year Treasury yield flat. Companies could also offer employees an inflation hedge by crediting at the CPI plus 300 basis points. The option is attractive for companies that believe that the real rates on inflation-linked Treasuries (currently around 4%) will be sustained.

Practice seems to focus on crediting at a rate given by the guidelines (perhaps for regulatory reasons not covered here), but it is interesting to note that the IRS Notice 96-8 says that the crediting rate is to be *no greater* than one of the rates given by a guideline. It may seem counterintuitive that the IRS is protecting employees by limiting the return on their pensions. After all, given today's cash balance, having a higher crediting rate implies more money for the worker either at retirement or at the time of an early departure. However, the rule that the IRS is enforcing is one that limits the loss of pension for workers who leave after an initial vesting period but before retirement. In other words, if we take as given the projected benefit at retirement, using a higher crediting rate implies a lower initial cash balance, and tends to penalize workers who leave the firm early, which disqualifies the plan for special tax considerations. Crediting at a rate less than one given by the guidelines is probably less expensive for the plan sponsor for a given level of employee satisfaction, since it is probably easier to communicate to employees the value of a higher income-related credit than the value of a higher interest-related credit.

The following example illustrates the cost and effective duration of two cash-balance liabilities, ignoring the complexities associated with uncertain future interest rates and compounding. The example compares two cash-balance liabilities with different crediting rates but the same current cash balance B_0 and time-to-exit of 4 years. Liability A credits at the 2-year zero-coupon bond rate and Liability B credits at the 3-year zero-coupon bond rate.

The cost of the liability depends on anticipated future interest rates. Consider the forward rates implied by the current term structure as forecasts of future interest rates. With no uncertainty, these forward rates would be perfect projections (or else there would be arbitrage). Denote by z_1, z_2, z_3, and z_n, the zero-coupon interest rates quoted now at time 0 for Treasury STRIPs maturing respectively one, two, three, and n years from now. Then, the one-year forward rates f_1, f_2, f_3, f_n implicit in today's yield curve for maturities 1, 2, 3, and n years from now are given approximately by solving $z_1 = f_1$, $z_2 = (f_1 + f_2)/2$, $z_3 = (f_1 + f_2 + f_3)/3$, ..., and $z_n = (f_1 + f_2 + f_3 + ... + f_n)/n$ for f_1,...,f_n. (This is only approximate because compounding is ignored. The overall approximation is actually better than it might seem because it is missing compounding in discounting as well as accumulation and the errors tend to cancel.) Therefore, $f_1 = z_1$, $f_2 = 2z_2 - z_1$, $f_3 = 3z_3 - 2z_2$, ..., and $f_n = nz_n - (n-1)z_{n-1}$.

The future crediting rates are inferred from the forward rates. Recall that Liability A credits at the 2-year zero-coupon yield and Liability B credits at the 3-year zero-coupon yield. Then, the crediting rate is $r_t^c = (f_t + f_{t+1})/2$ in Liability A and $r_t^c = (f_t + f_{t+1} + f_{t+2})/3$ in Liability B. Therefore, the terminal cash balance in Liability A is

$$
\begin{aligned}
(1) \quad B_4^A &= B_0 \left(1 + \frac{f_1 + f_2}{2} + \frac{f_2 + f_3}{2} + \frac{f_3 + f_4}{2} + \frac{f_4 + f_5}{2}\right) \\
&= B_0 \left(1 + \frac{f_1}{2} + f_2 + f_3 + f_4 + \frac{f_5}{2}\right),
\end{aligned}
$$

and the terminal cash balance in Liability B is

$$
\begin{aligned}
(2) \quad B_4^B &= B_0 \left(1 + \frac{f_1 + f_2 + f_3}{3} + \frac{f_2 + f_3 + f_4}{3} + \frac{f_3 + f_4 + f_5}{3} \right. \\
&\quad \left. + \frac{f_4 + f_5 + f_6}{3}\right) \\
&= B_0 \left(1 + \frac{f_1}{3} + \frac{2f_2}{3} + f_3 + f_4 + \frac{2f_5}{3} + \frac{f_6}{3}\right).
\end{aligned}
$$

The cost of the liability is the present value of the projected retirement balance, which is discounted back at the rates implicit in the yield curve. For Liability A, which credits at the 2-year zero-coupon rate, the cost is

$$
\begin{aligned}
(3) \quad C_0^A &= B_0 \frac{1 + \frac{f_1}{2} + f_2 + f_3 + f_4 + \frac{f_5}{2}}{1 + f_1 + f_2 + f_3 + f_4} \\
&\approx B_0 \left(1 + \left(\frac{f_1}{2} + f_2 + f_3 + f_4 + \frac{f_5}{2}\right) - (f_1 + f_2 + f_3 + f_4)\right) \\
&= B_0 \left(1 + \frac{1}{2}(f_5 - f_1)\right).
\end{aligned}
$$

Similarly, for Liability B, which credits at the 3-year zero-coupon rate, the cost is

$$(4) \quad C_0^B \approx B_0\left(\left(1+\frac{f_1}{3}+\frac{2f_2}{3}+f_3+f_4+\frac{2f_5}{3} + \frac{f_6}{3}\right)-(f_1+f_2+f_3+f_4)\right)$$
$$= B_0\left(1+\frac{2}{3}(f_5-f_2)+\frac{1}{3}(f_6-f_1)\right).$$

The deviation of the cost of the liability from the cash balance in the example depends on the slope of the yield curve and is due to a mismatch between the duration of the crediting asset and the interval between crediting rate resets. If the yield curve is flat, or if the crediting rate is the one-year yield, then the cost of the liability equals the cash balance. When the maturity of the crediting asset is greater than one year, the cost per unit of cash balance is one plus a weighted sum of differences between forward rates at long maturities and forward rates at short maturities. Given an upwardly sloping yield curve, accrual to the end is at higher rates than discounting back to the present, and the cost exceeds the cash balance. The opposite is true in a downwardly sloping yield curve, and the cost is less than the cash balance. These effects become more pronounced as the mismatch between the crediting period and the maturity of the crediting asset increases, as can be seen from comparing the formulas for liabilities crediting at 2-year and 3-year maturity bond rates. In a humped yield curve, like the benchmark yield curve used in the more exact computations in Section 2, the sign of the effect will vary with the length of the program and the duration of the crediting asset.

To see how this works, consider the two sample yield curves in Table 2, both of which are upwardly sloping, but one of which has a larger slope than the other. Starting from the currently quoted discount bond yields, forward rates and the two crediting rates are computed as described above.

With an upwardly sloping term structure and a crediting asset maturity greater than the one-year spacing between resets, the rates at which the cash balance credits are larger than the corresponding one-year forward rates

Table 2: A Numerical Example

Upwardly Sloping Yield Curve

Years Out	Zero-Coupon Yield Now	Forward Rate	2-year Cred. Rate	3-year Cred. Rate
1	5.0%	5.0%	5.2%	5.4%
2	5.2%	5.4%	5.6%	5.8%
3	5.4%	5.8%	6.0%	6.2%
4	5.6%	6.2%	6.4%	6.6%
5	5.8%	6.6%	NA	NA
6	6.0%	7.0%	NA	NA

Steeply Upwardly Sloping Yield Curve

Years Out	Zero-Coupon Yield Now	Forward Rate	2-year Cred. Rate	3-year Cred. Rate
1	5.0%	5.0%	5.5%	6.0%
2	5.5%	6.0%	6.5%	7.0%
3	6.0%	7.0%	7.5%	8.0%
4	6.5%	8.0%	8.5%	9.0%
5	7.0%	9.0%	NA	NA
6	7.5%	10.0%	NA	NA

which are the one-year bond returns that can be locked in today. Consequently, the cash balance grows faster than the corresponding rate at which future cash flows are discounted, and the value of the liability is larger than the cash balance. The amount by which the crediting rate exceeds the corresponding spot rate depends most significantly on any margin (none here), the slope of the yield curve (larger in the second panel of Table 2), and the duration of the bond underlying the crediting rate (larger for the three-year bond than for the two-year bond).

This example can also be used to highlight how the relative cost of liabilities depends on any margins added. In the upwardly sloping yield curve in the first panel, it would be cheaper to fund a liability that credits at the two-year rate unless the margin to be added to the two-year yield is more than 0.20% higher than the margin added to the three-year rate. With the steeper yield curve in the second panel, it would be cheaper to fund a liability that credits at the two-year rate unless the margin added to the two-year yield is more than 0.50% higher than the margin added to the three-year rate. It is also worth noting that when the yield curve is upwardly sloped, positive margins may be unappealing from the standpoint of the plan provider in that the cash balance grows faster than the bond returns even without margins.

The factors that determine the effective duration of a cash-balance liability can also be seen from (3) and (4). The effective duration of a cash-balance liability (or any other cash flow stream) is defined to be the maturity of a zero-coupon bond (like a Treasury STRIP) that has the same sensitivity as the cost to a shock in interest rates. A parallel shift in the term structure of interest rates implies that all spot rates and hence all forward rates change by the same amount. In this case, it is clear that the cost of the liability is invariant to interest rate changes and the effective duration is zero. The change in the expected accrual rates resulting from a rate change is exactly offset by the change in the discount factor, at least for a parallel shift in the yield curve.

More realistically, it is reasonable to expect that a shock to interest rates will have less impact on longer-term rates than on shorter-term rates. For example, the empirical estimation in Dybvig [1997] suggests that an interest rate shock loses about 15% of its impact for each year (1.26% each month) one moves out the forward rate curve. Market participants are likely to react to news today by making large changes in their views about interest rates over the next year but they are less likely to make large changes in their views about interest rates twenty-five years from now. (Consistent with this idea, the interest rate volatilities implicit in bond option contracts tend to decrease with the maturity of the underlying bond.)

The normal pattern is that an event that results in large increases in forward rates one and two years out, f_1 and f_2, will generate smaller increases in forward rates four and five years out, f_4 and f_5. This pattern will cause the cost of the liability that accrues at the three-year rate to go down, as can be seen from (4). This implies that the effective duration is greater than zero, since the cost moves in the same direction as a bond price would. The duration is, however, much less than the maturity of the liability, due to the cancellation of rates in the derivation of the pricing formulas. For example, a zero-coupon bond with two years to maturity, which by definition has a duration of 2, would have a price of about $B_0(1-f_1-f_2)$ and would be more sensitive to an interest rate move than either cash-balance liability maturing in 4 years. It is also clear from (3) and (4) that Liability B, which credits at the three-year rate, is more sensitive to interest rates, and therefore has a higher effective duration, than the liability that accrues at the two-year rate.

2 Computed Cost and Effective Duration of Various Liabilities

This section evaluates the cost of funding a cash-balance liability given a particular benchmark yield curve. Analysis here includes all the underlying rates in Table 1 except for the CPI. (It is hard to perform an objective comparison of the CPI case with the others because a primary determinant in the comparison would be beliefs about the future connection between interest rates and inflation.) For each underlying rate, both the cost and the effective duration are computed (a) for no margin ("flat") and for the margin under the IRS guideline, (b) for three liability maturities of 10 years, 20 years, and 30 years, and (c) for both random and nonrandom interest rate models. Each number computed for the random interest rate model is based on Monte Carlo simulation with 200,000 random interest rate paths, implying a simulation error of no more than a few basis points.

The model of interest rates used for computing the terminal balance and discounting back to the present is critical for computing the cost. The analysis is based on one of two interest-rate models consistent with a benchmark Treasury STRIP curve traded for settlement on November 15, 1999. (Choosing November 15 was convenient because the STRIPs mature at equal three-month intervals from that date.) The simpler model is a certainty model which forecasts that future interest rates will be equal (for certain) to a smoothed and extrapolated version of the implied forward rates. (Smoothing removes economically meaningless jumping around of forward rates that is likely due to mispricing within the bid-asked spread. Extrapolation beyond the last STRIP is required for rates beyond the end of the current STRIP curve. If a program has 20 years to maturity and credits at a 30-year rate, computing the cost requires implied forward rates out 50 years. The smoothing is based on a least-squares fit of a quintic to the implied forward-rate curve, constrained to be flat at the end, extrapolating smoothly using a flat forward rate curve beyond the set of available STRIP prices.) This certainty model is like the analysis of the extended example in the previous section but it includes proper accounting for compounding and other institutional details. The more sophisticated model describes the uncertainty in interest rates using a mean-reverting Vasicek (Gaussian) model of interest rates in which the mean rates are assumed to vary over time in a way that replicates the smoothed and extrapolated version of the yield curve on November 15, 1999. The standard deviation of interest rates is taken to be 1% per year, and the mean reversion in the model is taken to be 15% per year, consistent with actual observed yield curves. (See Dybvig [1997] for more on the Vasicek model with "fudge factors" to fit a given yield curve and also for estimates of volatility and mean reversion.)

The calculations of cost and effective duration are shown in Tables 3 and 4. Table 3 gives the results from the sophisticated model with random interest rates, and Table 4 gives the results from the simpler certainty model. Within each table, the top panel gives the cost when crediting at a given rate without any margin, and the bottom panel gives the cost when crediting at the rate plus the IRS margin given in Table 1. Within each panel, the maturity of the program is given in the top row, and the base crediting rate is given in the left column. The numbers in the table depend on the specific yield curve and implementation details such as the frequency of compounding or whether crediting is at the average rate over the previous year or at the rate at the start of the year. However, the main conclusions do not seem to rely on these details.

IRS Margins are Important and Indeed Very Expensive

Perhaps the most dramatic pattern in the Tables is the impact of the IRS margins. From Table 1, there are no margins for crediting at 10-year, 20-year, or 30-year rates. However, for all other underlying rates, the IRS margin has the dominant effect on cost. For example, the cost per dollar of cash balance in the IRS case for 20-year maturity and crediting at the 1-year yield is \$1.202. The margin of \$0.202 corresponds to a compounded return of 92.42 basis points per year, which can be viewed as the excess yield of the liability over the yield of Treasuries of the same effective duration. This excess is attributable almost entirely to the 100-basis-point margin in the IRS rules.

One way to think of the impact of the IRS guidelines is to ask how much the slope of the yield curve would have to change to overcome the impact of the margin. Under the IRS guidelines, crediting based on the one-year Treasury yield is associated with a margin of 100 basis points but crediting at the 30-year Treasury yield has no associated margin. In the benchmark case given in the second panel of Table 3 with 20 years to maturity, the cost of the liability is \$1.181 with crediting at the one-year yield or \$0.920 with crediting at the 30-year yield. The difference in cost is \$0.261. To compute how much to change the slope of the yield curve to make the two crediting rates equally costly, the rule-of-thumb developed later in the paper ((13) and (14)) can be used. Based on this rule, the cost under the one-year yield does not vary with the slope of the yield curve (since there is no mismatch between the maturity of the crediting asset and the reset period). Taking the duration of a 30-year bond to be 9 years (this exact value is not critical), then the required change in the difference between the 20-year forward rate and the spot rate is (by (13)) equal to a whopping 6.5% ($= 0.261/((9-1)/2)$).

From a plan sponsor's perspective, there are strong incentives to choose a crediting alternative without any margin (if choosing according to the guide-

Table 3: Market-value cost and effective duration of cash-balance liabilities: random model without any margins added to the market rates. These calculations are for liabilities with known maturity and no early departure before maturity. The cost is per dollar of cash balance, and can deviate significantly from par. Because the cash balance credits at market yields, its effective duration is significantly less than the time to maturity in all cases.

no margins	10yr maturity		20yr maturity		30yr maturity	
random model	cost	effdur	cost	effdur	cost	effdur
3mo discount	0.962	0.599	0.930	1.046	0.905	1.321
6mo discount	0.959	0.754	0.923	1.315	0.894	1.664
12mo discount	0.955	1.061	0.907	1.847	0.871	2.342
1yr yield	0.990	0.247	0.979	0.428	0.971	0.537
2yr yield	0.998	0.399	0.983	0.678	0.973	0.858
3yr yield	1.004	0.546	0.985	0.921	0.973	1.169
5yr yield	1.013	0.822	0.985	1.376	0.972	1.758
7yr yield	1.018	1.072	0.982	1.791	0.968	2.299
10yr yield	1.020	1.395	0.973	2.338	0.958	3.025
20yr yield	1.003	2.113	0.939	3.650	0.923	4.823
30yr yield	0.989	2.542	0.920	4.491	0.904	6.003

The same model changes dramatically when IRS margins are added.

IRS margins	10yr maturity		20yr maturity		30yr maturity	
random model	cost	effdur	cost	effdur	cost	effdur
3mo discount	1.133	0.727	1.290	1.266	1.479	1.606
6mo discount	1.104	0.863	1.222	1.503	1.362	1.907
12mo discount	1.099	1.166	1.202	2.030	1.329	2.582
1yr yield	1.087	0.323	1.181	0.557	1.286	0.704
2yr yield	1.045	0.437	1.079	0.742	1.120	0.941
3yr yield	1.052	0.584	1.082	0.985	1.120	1.252
5yr yield	1.037	0.841	1.033	1.408	1.043	1.799
7yr yield	1.042	1.090	1.029	1.822	1.038	2.340
10yr yield	1.020	1.395	0.973	2.338	0.958	3.025
20yr yield	1.003	2.113	0.939	3.650	0.923	4.823
30yr yield	0.989	2.542	0.920	4.491	0.904	6.003

Table 4: Market-value cost and effective duration of cash-balance liabilities: certainty model without any margins added to the market rates. These calculations are for liabilities with known maturity and no early departure before maturity. The cost is per dollar of cash balance, and can deviate significantly from par. Because the cash balance credits at market yields, effective duration is significantly less than the time to maturity in all cases.

no margins	10yr maturity		20yr maturity		30yr maturity	
certainty model	cost	effdur	cost	effdur	cost	effdur
3mo discount	0.963	0.596	0.935	1.038	0.913	1.305
6mo discount	0.961	0.752	0.928	1.307	0.902	1.647
12mo discount	0.956	1.059	0.913	1.838	0.880	2.325
1yr yield	0.991	0.246	0.981	0.423	0.974	0.530
2yr yield	0.998	0.398	0.984	0.674	0.974	0.851
3yr yield	1.004	0.545	0.985	0.917	0.974	1.163
5yr yield	1.012	0.821	0.984	1.372	0.972	1.752
7yr yield	1.017	1.070	0.981	1.787	0.967	2.295
10yr yield	1.019	1.394	0.971	2.335	0.957	3.021
20yr yield	1.002	2.115	0.937	3.651	0.922	4.824
30yr yield	0.988	2.546	0.918	4.495	0.903	6.007

The same model changes dramatically when IRS margins are added.

IRS margins	10yr maturity		20yr maturity		30yr maturity	
certainty model	cost	effdur	cost	effdur	cost	effdur
3mo discount	1.135	0.726	1.297	1.259	1.493	1.592
6mo discount	1.106	0.861	1.228	1.495	1.375	1.892
12mo discount	1.101	1.165	1.209	2.023	1.343	2.567
1yr yield	1.088	0.322	1.183	0.553	1.290	0.697
2yr yield	1.046	0.436	1.080	0.738	1.122	0.934
3yr yield	1.052	0.582	1.082	0.980	1.122	1.246
5yr yield	1.036	0.839	1.032	1.404	1.043	1.793
7yr yield	1.041	1.088	1.028	1.818	1.038	2.335
10yr yield	1.019	1.394	0.971	2.335	0.957	3.021
20yr yield	1.002	2.115	0.937	3.651	0.922	4.824
30yr yield	0.988	2.546	0.918	4.495	0.903	6.007

lines), even if it is believed that investment performance can beat the margin in the IRS guidelines. For example, by taking on some acceptably small amount of credit risk, a plan sponsor might think it is easy to achieve the 1-year Treasury yield plus 100 basis points. However, by crediting at the 10-year rate, the plan sponsor could take on the same credit risk, lengthen the portfolio a bit to re-align the duration, and collect the same 100 basis points to reduce future contributions. Conceptually, it is possible that a firm might be able to explain to workers the benefit of a short-maturity crediting rate and the corresponding margin, but it is more likely that whatever value is added for the workers will not be appreciated.

The dominant impact of the IRS margins could, in principle, be overcome by other factors in extreme cases, for example in a very steep upwardly-sloping yield curve. However, this sort of temporary condition should not usually affect the ranking of crediting rate alternatives. While the yield curve today may make it attractive to choose an IRS option with a margin for contributions today, the yield curve will change next year, in 5 years, and in 10 years, and the plan sponsor is likely to lose more on future contributions and new employees than is gained on current plan members' current cash balances. The only possible exception may be in a plan of inactive participants without ongoing contributions; even so, it will rarely be optimal to take on a margin.

From a social policy perspective, IRS guidelines that encourage crediting at long rates may not be a good idea. Even if yields at the long end are expected to be higher, crediting at the long rate is riskier in real terms for plan participants, since short rates track inflation better than long rates. Furthermore, crediting at a long rate is riskier for the Pension Benefit Guarantee Corporation (PBGC). A fund crediting at a long rate and using the theoretical duration hedge is subject to a sort of reverse whip-saw effect, namely that underlying investments move around more than the cash balance itself. If the rates go up and the assets' value goes down, the plan may not have enough assets to liquidate to cover a large number of departures. This is only a problem for the pension insurance fund if it happens at a time when the firm is unable to make up the difference, but this may happen broadly in the economy in a time of high interest rates and high unemployment, thus creating big problems for the PBGC's insurance fund.

Because the CPI is not a market rate, the 300-basis-point margin endorsed by the IRS is not necessarily excessively costly. Crediting at the CPI plus 300 basis points may be appealing to workers (because the final payment is not risky in real terms) and to employers as well (because the margin is less than at least the current yield on indexed Treasuries). Computing the cost and effective duration for a cash-balance liabilitity credited at CPI plus 300 basis points is more difficult than for the other cases and necessarily depends significantly on a view of what inflation will be and how it will relate to interest rates.

A Cash-Balance Pension Liability's Effective Duration is Typically Positive but Smaller than the Maturity of the Liability or the Duration of the Asset Underlying the Crediting Rate

The specification of a cash-balance liability includes several time variables: time to maturity of the liability, duration of the assets underlying the crediting rate, and time between resets of the crediting rate. One might hope that one of these times would be the effective duration of the liability, but the tables reveal a more subtle pattern. Examining why different likely candidates are not the correct duration of the liability is a good way to develop intuition for how the effective duration is determined.

The effective duration is not the time to maturity of the cash-balance liability. If a fixed amount were to be paid at the end, this would be the duration, but the amount paid is indexed to interest rates, and this indexing offers some protection against interest-rate risk. That is, the claim is less sensitive to interest-rate risk than a fixed claim at the same date, implying a shorter effective duration.

The effective duration is positive, at least so long as the duration of the crediting asset is longer than the time between resets. Although one might conclude that the indexing provides full insurance against interest rate movements, implying a duration of zero, longer rates move less than the short rate and therefore insurance is partial and the effective duration is typically positive.

The effective duration is not (except by accident) equal to the duration of the crediting asset. Crediting using a discount rate (a Treasury Bill's discount to face value as a proportion of face value, divided by the fraction of a fictional 360-day year to maturity) can have some peculiar properties (since the discounts themselves move less than the corresponding yields). However, crediting using yields always implies an effective duration less than the duration of the crediting asset. It might be tempting to think of the cash balance itself as being the valuation of a constant-maturity portfolio invested in the asset underlying the crediting rate. However, because the cash balance is credited the yield but does not experience any capital gains or losses, it is less sensitive to interest rates than a corresponding constant-maturity portfolio.

The effective duration is not (except by accident) equal to the time between rate resets. One effect ignored here is the change of duration between rate resets, because duration in our analysis is always measured at a point in time just before a rate reset. When the time between resets is equal to the duration of the crediting asset and is also equal to the compounding interval, then the effective duration is zero. Between resets, the duration would be the time until the next reset. Except in this special case, there is no reason that duration would equal the time until the next reset.

Hopefully, these observations are at least suggestive of the subtleties of the determination of the effective duration. For crediting rates based on yields, the effective duration increases with the maturity of the liability and the maturity of the crediting asset. For all of these cases, the effective duration is less than either the maturity of the liability or the maturity of the crediting asset. For crediting based on Treasury discounts, durations can be larger (because discounts move less than yields), but the results are otherwise similar. In general, the duration changes little when the IRS margin is added.

The Slope of the Yield Curve Affects the Cost

Although not evident from the tables (since they are all based on the same yield curve), the slope of the yield curve does affect the cost of the liability. As illustrated in the example in Section 1, crediting at a long-maturity rate in an upward-sloping yield curve implies a cost greater than the cash balance. In the yield curve of November 15, 1999, the implied forward rates rise until about 10 years out and then fall to near the starting value about 30 years out. This explains why, of the liabilities without any margins, the 10-year liabilities have the highest cost and the 30-year liabilities are the cheapest.

Modeling the Uncertainty is Not So Important

One pleasant surprise is that the results in Tables 3 and 4 generally do not differ by much. In terms of the theoretical literature on the term structure, this is because the convexity is not so important. That is because of the cancellation of movements in the crediting rate and the discount rate as was evident in the example in Section 1: as interest rates rise, the increase in the discount rates is offset by an increase in the crediting rates. (Convexity *can* matter a lot in valuation of long-maturity fixed-income obligations, as emphasized by Dybvig and Marshall [1996].)

Other Factors Affecting Cost and Effective Duration

It has already been mentioned that the yield curve has a significant impact on cost; for this reason alone it should be clear that the results in Tables 3 and 4 cannot be applied directly for liabilities at any date besides November 15, 1999. Some of the choices in the simulation are based on industry-standard conventions, for example, computing bond yields using semi-annual compounding or computing Treasury discounts using the customary formula based on a fictitious 360-day year. Other choices are more arbitrary because there is no clear convention in practice, and changing these implementation details can affect the results. For example, using the average of rates over the last year instead of the quoted rate at the start of the year (as is used in the simulations) might change the cost and duration, especially when using a Treasury discount for the underlying rate. The choice of compounding interval for the liability will also affect the cost of the liability. The Tables here are based on an intermediate case of quarterly compounding. If the liability compounded annually, the cost would be lower, while if the liability compounded continuously, the cost would be higher. Finally, recall that by definition, the cost excludes the impact of early departures from the liability and any future contributions.

3 Cost of Cash-Balance Liabilities

Option pricing theory provides a reliable approach to evaluating the cost of cash-balance liabilities. This approach gives formulas for discrete-time models or continuous-time models. Both should give similar answers, and the choice between using discrete time and continuous time seems to be a matter of convenience. Discrete time is handy for working with the available data (although fractional periods can be a nuisance), while continuous time simplifies compounding and tends to give us simpler exact formulas. This paper provides discrete time formulas with occasional reference to continuous time.

Consider a liability with with crediting rate r_t^c. Then, absent any withdrawals or pay-related credits, the cash balance B_t satisfies

$$(5) \quad B_t = B_{t-1}(1 + r_t^c)$$

or equivalently

$$(6) \quad B_t = B_0(1 + r_1^c)(1 + r_2^c)(1 + r_3^c)...(1 + r_t^c).$$

These equations assume that the crediting rate corresponds to the time interval for compounding. For example, if going from $t = 0$ to $t = 1$ is a year, r_t^c is an annual rate, while if going from $t = 0$ to $t = 1$ is half a year, then r_t^c is half the annual rate.

Modern financial theory says that in the absence of arbitrage, cash flows are valued using expected present values computed using discounting at the spot rate and expectations using artificial risk-neutral probabilities (see, for example, Dybvig and Ross [1987]). If these probabilities are the actual probabilities, then the *local expectations hypothesis* holds and all assets have the same expected return. Otherwise, the risk-neutral probabilities are artificial probabilities that undo the risk premia. For pricing interest derivatives, it does not matter so much which assumption is made so long as (1) the model for interest rates is consistent with the observed yield curve and (2) the volatility of interest rates is modelled well. Whichever the case, the expectation $E[\cdot]$ is taken to be the appropriate expectation for valuation. Present values are computed using the rolled-over spot rate and therefore the cost C_0 of the liability with initial balance B_0 and horizon T is

$$
\begin{aligned}
(7) \qquad C_0 &= E\left[\frac{B_T}{(1+r_1)(1+r_2)(1+r_3)...(1+r_T)}\right] \\
&= E\left[B_0\frac{(1+r_1^c)(1+r_2^c)(1+r_3^c)...(1+r_T^c)}{(1+r_1)(1+r_2)(1+r_3)...(1+r_T)}\right],
\end{aligned}
$$

where r_t is the spot interest rate quoted at $t-1$ for payment at t (but is not the corresponding forward rate which is stochastic).

It is apparent from (7) that the cost equals the initial cash balance if the crediting rate equals the spot rate $r_t = r_t^c$: in this case, each term in the numerator cancels the corresponding term in the denominator and therefore $C_0 = E[B_0] = B_0$. It is slightly more subtle to see that if the crediting rate is a zero-coupon bond yield for a bond maturing at the next reset, the cost equals the initial cash balance in this case as well. This is because the zero-coupon bond yield d_t^{t+M} quoted at time t for maturity $t+M$ satisfies

$$
(8) \qquad \frac{1}{(1+d_t^{t+M})^M} = E_t\left[\frac{1}{(1+r_{t+1})(1+r_{t+2})...(1+r_{t+M})}\right]
$$

or

$$
(9) \qquad 1 = E_t\left[\frac{(1+d_t^{t+M})(1+d_t^{t+M})...(1+d_t^{t+M})}{(1+r_{t+1})(1+r_{t+2})...(1+r_{t+M})}\right]
$$

where $E_t[\cdot]$ indicates expectation conditional on information at time t. Crediting at the zero-coupon bond yield over the period from t to $t+M$ means ratios of corresponding terms in the numerator and denominator of (7) will have expectation 1, and from this observation it is not too difficult (using the law of iterated expectations) to complete a formal proof that crediting the zero-coupon bond yield corresponding to the time between resets gives cost equal to cash balance.

To use (7) when interest rates are random, both the spot rate and the crediting rate are simulated using a Vasicek [1977] term structure model with means adjusted to fit today's yield curve (as described by Heath, Jarrow, and Morton [1992] or more explicitly by Dybvig [1997]). Then, the appropriate coupon bond yield can be derived from the simulated forward rate curve using a formula such as those given in Table 1 of Dybvig, Ingersoll, and Ross [1996]. Such an approach gives the most accurate valuation, provided the simulation has enough random draws (100,000 to 1,000,000 draws are typically enough for the required precision and do not take too long on modern computers).

An alternative to using random interest rates is to assume interest rates are not random and that the actual future interest rates are the forward rates implicit in today's Treasury STRIP curve. This is obviously not literally true, but it simplifies the problem considerably. In practice, it requires a little finesse to interpolate and extrapolate for maturities not observed, and to smooth values made ragged because of bid-ask effects or asynchronous quotes, but the basic idea is as follows. Take as given observations at time 0 of the zero-coupon bond prices D_0^t for all t. Then, the implied forward rates f_0^t are given by the standard formula

$$(10) \quad D_0^t = D_0^{t-1} \frac{1}{1+f_0^t},$$

or

$$(11) \quad f_0^t = \frac{D_0^{t-1}}{D_0^t} - 1$$

since the absence of arbitrage implies the price is the same for a cash flow at time t whether it is bought directly (which has price D_0^t) or indirectly by buying cash at time $t-1$ (at price D_0^{t-1}) to invest at the forward rate until t (at a price of $1/(1+f_0^t)$ at time $t-1$ for each dollar of cash at time t). Assuming interest rates are nonrandom, $r_t = f_0^t = f_s^t$, for all $s < t$, i.e., borrowing or lending from $t-1$ to t is the same whether contracted at $t-1$ (at rate r_t), at 0 (at rate f_0^t), or at s (at rate f_s^t). Otherwise, borrowing or lending forward with the offsetting trade in the spot market would be an arbitrage. Thus, the forward rate curve has all the information needed to compute both the terminal balance B_T and the cost C_0.

A Nifty Rule-of-Thumb

A useful rule-of-thumb generalizes the example in Section 1. Take as given annual forward rates f_0^1, f_0^2,... computed from today's Treasury STRIP curve. Assume the crediting rate is a 3-year zero-coupon bond yield and assume that the maturity T is much larger than 3. Then, noting that compounding over a few periods is not so important, we invoke the approximation $(1+x)(1+y) \approx 1+x+y$ for x and y small. Therefore the crediting rate at t is $r_t^c = ((1+f_0^t)(1+f_0^{t+1})(1+f_0^{t+2}))^{1/3} - 1 \approx (f_0^t + f_0^{t+1} + f_0^{t+2})/3$ and consequently

$$
\begin{aligned}
(12)\quad C_0 &= B_0 \frac{(1+r_1^c)(1+r_2^c)(1+r_3^c)...(1+r_T^c)}{(1+r_1)(1+r_2)(1+r_3)...(1+r_T)} \\
&\approx B_0 \frac{(1+f_0^1/3)(1+2f_0^2/3)(1+f_0^3)...(1+f_0^T)(1+2f_0^{T+1}/3)(1+f_0^{T+2}/3)}{(1+r_1)(1+r_2)(1+r_3)...(1+r_T)} \\
&= B_0 \frac{(1+2f_0^{T+1}/3)(1+f_0^T/3)}{(1+2f_0^1/3)(1+f_0^2/3)} \\
&\approx B_0(1+f_0^T - f_0^1),
\end{aligned}
$$

where most of the approximations ignore compounding, and the last one also assumes that the yield curve is reasonably flat at the beginning and the end. Intuitively, you are credited with all the rates in the middle, although partly displaced by a period or two. Therefore, all the middle rates cancel with the interest rates in the denominator, and the result is based on the difference between rates at the beginning and end. More generally, a cash balance liability credited at an M-period discount yield, adjusted every period, with a maturity $T > M$, has a cost of approximately

$$(13)\quad C_0 = B_0 \left(1 + \frac{M-1}{2}(f_0^T - f_0^1)\right),$$

where $(M-1)/2$ can be interpreted as half the mismatch of maturities between reset and crediting, and $f_0^T - f_0^1$ can be interpreted as the slope of the (forward rate) yield curve. While not appropriate for critical applications requiring an exact number, this simple rule works surprisingly well.

The rule-of-thumb (13) was derived under the assumption that there is no margin (IRS-prescribed or otherwise). Given a margin of π per unit time, the liability is larger by the factor $(1+\pi)^T$, and (13) becomes

$$(14)\quad C_0 = B_0 \left(1 + \frac{M-1}{2}(f_0^T - f_0^1)\right)(1+\pi)^T.$$

4 Effective Duration of Cash-Balance Plans

To compute the interest-rate exposure of any fixed-income security or derivative, the starting point must include some view, implicit or explicit, of how interest rate shocks today affect future rates. Analysis in this paper uses the simple Vasicek model with mean reversion (with or without adjustments to fit the initial yield curve). While this model is specialized and too simplistic for some purposes, it serves well in this context. In this model, a shock to today's interest rate is reflected linearly in future realized interest rates, with a declining effect over time. Specifically, a shock δ to the spot rate at time 0 has an impact $\delta \exp(-kt)$ on the spot rate at t. The parameter k is the rate of mean-reversion in interest rates; the larger k is, the more temporary is the impact of the shock and the smaller $\delta \exp(-kt)$ will be for each t. The effective duration is then the maturity of the zero-coupon bond with both the same cost and the same sensitivity to δ as the claim.

One feature of the Vasicek model that simplifies the task significantly is the following simple formula for the impact of a shock δ in the instantaneous interest rate on the zero-coupon bond price D_0^t.

$$(15)\quad \frac{dD_0^t}{d\delta} = -\frac{1-e^{-kt}}{k} D_0^t$$

and therefore solving for t obtains

$$(16)\quad t = -\frac{1}{k}\log\left(1 + k\frac{1}{D_0^t}\frac{dD_0^t}{d\delta}\right).$$

This expresses the effective duration of a zero-coupon bond in terms of its sensitivity to shock δ per unit of value. By the definition of effective duration (risk exposure per dollar of value is the same as for a zero-coupon bond with maturity at the claim's duration), the expression must be the same for all claims with duration t, and therefore

$$(17)\quad \begin{aligned} t &= -\frac{1}{k}\log\left(1 + k\frac{1}{C_0}\frac{dC_0}{d\delta}\right) \\ &= -\frac{1}{k}\log\left(1 + k\frac{d\log(C_0)}{d\delta}\right) \end{aligned}$$

is the effective duration of any liability whose cost is C_0. In the limit as k tends to 0, there is no mean reversion, and this formula becomes the same as what would be implied by the traditional Macauley duration based on parallel shifts of the yield curve.

There are a number of ways to compute this derivative for use in the effective duration formula (17). One simple way is to compute it numerically by considering directly shocks to the initial yield curve. Let D_0^t be implied from the initial given yield curve (perhaps a smoothed and extrapolated version of data on Treasury STRIPs). Then, picking some small number δ_0, consider the impact of positive and negative shocks this size on the the initial yield curve. Let

$$(18)\quad D^t_{0,up} \equiv D^t_0 + \delta_0 \left(-\frac{1-e^{-kt}}{k} D^t_0 \right)$$

and let

$$(19)\quad D^t_{0,down} \equiv D^t_0 - \delta_0 \left(-\frac{1-e^{-kt}}{k} D^t_0 \right).$$

Then, (15) and the definition of a derivative implies that

$$(20)\quad \frac{d\log(dC_0)}{d\delta} \approx \frac{C_{0,up} - C_{0,down}}{2\delta},$$

where $C_{0,up}$ is based on $D_{0,up}$ and $C_{0,down}$ is based on $D_{0,down}$. The costs $C_0(D_{0,up})$ and $C_0(D_{0,down})$ can be computed numerically, and then substituting (20) into (17) gives the effective duration. (If this computation uses a simulation, the same random draws should be used for the up and down cases or else convergence will be slow.) This is the approach that was used to compute the effective duration numbers in Table 3.

5 Conclusion

This analysis of the market-value cost and effective duration of various cash-balance pension liabilities has considered term structure effects and a variety of crediting rules using market rates with or without margins given by IRS guidelines. The general theory included here can be adapted to match specific institutional details that may differ from the assumptions here, or to include a model incorporating random departures of beneficiaries.

References

Anand, Vineeta, 1999. "Consultants Seek Cash Balance Legislation." *Pensions and Investments*, January 11.

Coleman, Dennis R., 1998. "The Cash Balance Pension Plan." *The Pension Forum* vol. 11, Society of Actuaries, 21–47.

Dybvig, Philip H., 1997. "Bond and Bond Option Pricing Based on the Current Term Structure." *Mathematics of Derivative Securities*, Michael A. H. Dempster and Stanley Pliska, eds., Cambridge University Press.

Dybvig, Philip H., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1996. "Long Forward and Zero-Coupon Rates Can Never Fall." *Journal of Business* vol. 69, 1–25.

Dybvig, Philip H., and William J. Marshall, 1996. "Pricing Long Bonds: Pitfalls and Opportunities." *Financial Analysts Journal*, January-February, 32–39.

Philip H. Dybvig and Stephen A. Ross, 1987. "Arbitrage." *The New Palgrave: a Dictionary of Economics* vol. 1, New York: Stockton Press, 100–106.

Heath, David, Robert Jarrow, and Andrew Morton, 1992. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation." *Econometrica* vol. 60, 77–105.

Kopp, Steve J., and Lawrence J. Sher, 1998. "A Benefit Value Comparison of a Cash-Balance Plan with a Traditional Final Average Pay Defined-Benefit Plan." *The Pension Forum* vol. 11, Society of Actuaries, 1–19.

Vasicek, Oldrich, 1977. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics* vol. 5, 177–188.