DERIVATIVE SECURITIES
Lecture 2: Binomial Option Pricing: Call, Put, and Futures Options

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- review of pricing formulas
- assets versus futures
- practical issues
- call options
- put options
- options on futures

Binomial asset pricing: review

Riskless bond: $\quad 1 \quad \longrightarrow \quad r$

Stock: $\quad 1 \quad \begin{array}{l} \nearrow \; u \\ \searrow \; d \end{array}$

Derivative security: $\quad ??? \quad \begin{array}{l} \nearrow \; V_u \\ \searrow \; V_d \end{array}$

$$Value = p_u V_u + p_d V_d = r^{-1}(\pi_u^* V_u + \pi_d^* V_d)$$

where

$$p_u = r^{-1}\frac{r-d}{u-d} \quad p_d = r^{-1}\frac{u-r}{u-d}$$

and

$$\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}.$$

Interpretation: all investments are fair gambles, subject to discounting to account for impatience and a probability adjustment to account for risk pricing.

Some orders of magnitude

- Expected excess returns
  - Common stock indices: 4–5% per year or 4–$5\%/250 \approx 1.5$ or 2 basis points daily
  - Individual common stocks: 50%–150% of the index expected excess return.
- Standard deviation
  - Common stock indices: 15–20% per year or 15–20%$/\sqrt{250} \approx 0.75$–1% per day
  - Individual common stocks: 35%–40% per year

Theoretical observation: for the usual case, standard deviations over short periods are almost exactly the same in actual probabilities as in risk-neutral probabilities.

Digression: review of interest rates and compounding

Simple interest: the amount paid for use of money is Prt, where P is principal, r is the interest rate, and t is the time interval (most commonly measured in years). For example, the interest on $\$200$ for 2 months at a 12% annual rate is $\$200 \times 12\% \times 2/12 = \$4$.

Compound interest: interest is recorded periodically and added to the principal; consequently there is interest on the interest. For example, if 12% interest is compounded semi–annually and we start with $\$1000$, then after three years we will accumulate $\$1000 \times (1 + 12\%/2)^{2\times 3} \approx \$1418.52$.

Continuous compounding: the value in the limit as interest is recorded more and more frequently is given by the exponential function. For example, if 12% interest is compounded continuously and we start with $\$1000$, then after three years we will have $\$1000 \times \exp(12\% \times 3) = \$1000e^{0.36} \approx \$1433.33$.

Over short horizons, all three are almost exactly the same!

Binomial parameters in practice

Most texts seem to have unreasonably complicated expressions for u, d, and r in binomial models. From the theory, we know that a good choice is

$$r = 1 + r_f \Delta t$$

$$u = 1 + r_f \Delta t + \sigma\sqrt{\Delta t}$$

$$d = 1 + r_f \Delta t - \sigma\sqrt{\Delta t}$$

with $\pi_u^* = \pi_d^* = 1/2$ and Δt the time increment. This has the two essential features: it equates risk-neutral expected stock and bond returns, and it has the right standard deviation. In addition, it has a continuous stock price (like Black-Scholes) as a limit.

One alternative is to choose the positive solution of $ud = 1$ and $u - d = 2\sigma\sqrt{\Delta t}$ (good for option price as a function of time) or a recombining trinomial (good for including some dependence of variance on the stock price).

European call option

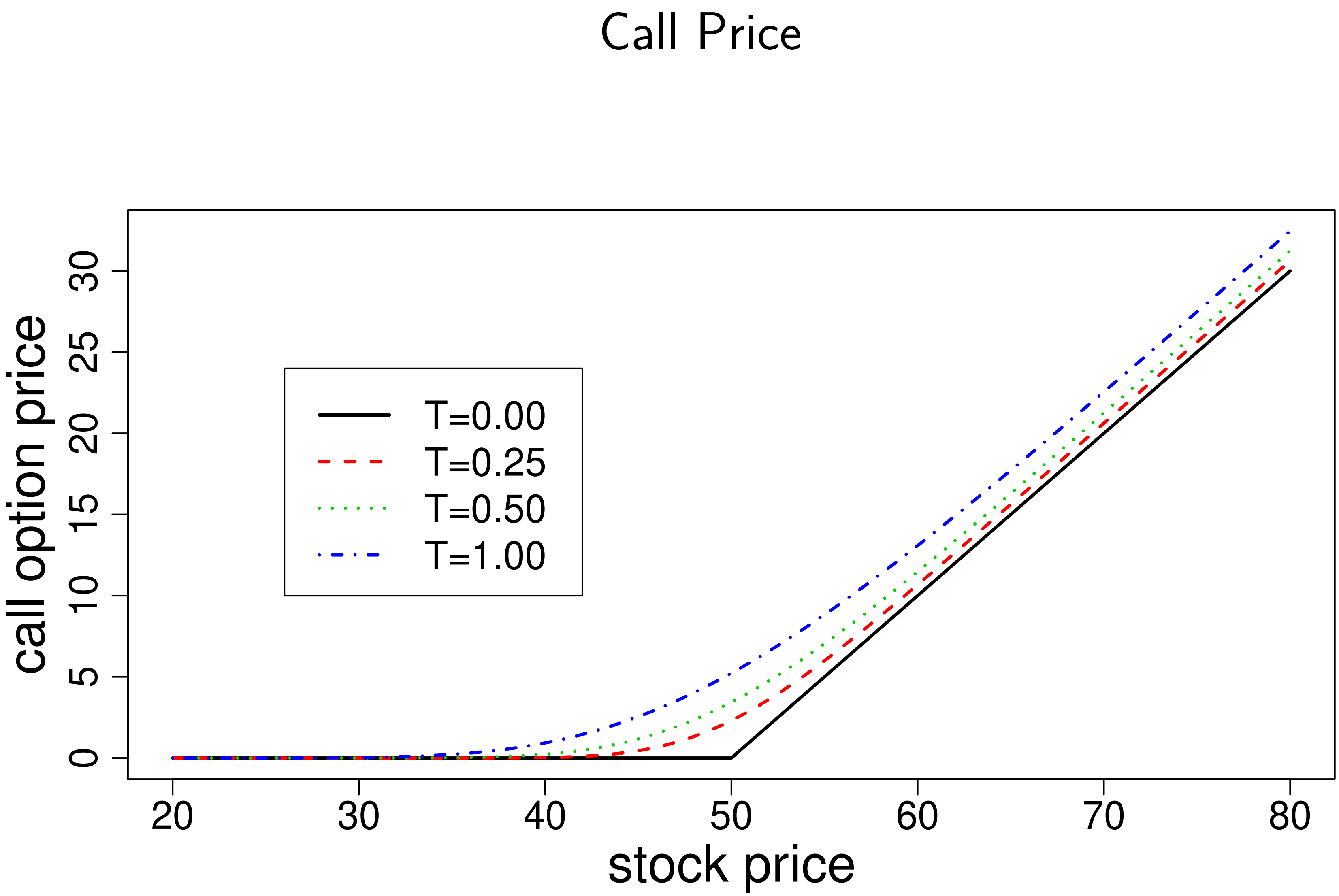
The purchaser obtains the right to buy one share of stock from the issuer at the maturity date for the pre-specified strike or exercise price.

If the stock price exceeds the strike price at maturity, the option is $in\ the\ money$. In this case, it is optimal for the owner (the purchaser) to exercise the option. The option is worth the difference between the stock price and the strike price, which is the amount the owner pockets by exercising the option.

If the strike price exceeds the stock price at maturity, the option is $out\ of\ the\ money$. In this case, it is optimal for the owner to let the option expire without exercising it, because exercise would cost the owner the difference between the stock price and the strike price. The price of the option is zero.

Summary: At maturity (T), the value is $\max(S_T - X, 0)$, or the maximum of the stock price less the exercise price and zero.

An $American\ call\ option$ is just like the European option, but the option of exercising is available anytime (once only) at or before the maturity date.


Call Price
call option price
stock price
T=0.00
T=0.25
T=0.50
T=1.00
0
5
10
15
20
25
30
20
30
40
50
60
70
80

More on call options

The call option pays off when the underlying stock goes up but does not obligate the owner when the underlying stock goes down. For this privilege, the purchaser pays a price (also called a *premium*) up front. The market price of the option depends on the exercise price, the stock price, the time to maturity, the volatility of the underlying stock, the riskless interest rate and the anticipated size of dividends before maturity.

The most important influence on an option's value day-to-day is the underlying stock's price. The next most important influence is changing volatility of the underlying stock price. In fact, one interesting use of option prices is to compute *implied volatilities*, that is the level of volatility ("vol") implicit in the option price. The other influences either do not change on a daily basis (the exercise price and the anticipated remaining dividends) or have movements with relatively minor effect on the option price at normal maturities (time to maturity and the riskless interest rate).

In-class exercise: call option valuation

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 1$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a call option with a \$90 strike price maturing in two periods?

stock

call option

reminder: $\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}$.

European put options

The purchaser obtains the right to sell one share of stock to the issuer at the maturity date for the pre-specified strike (or exercise) price.

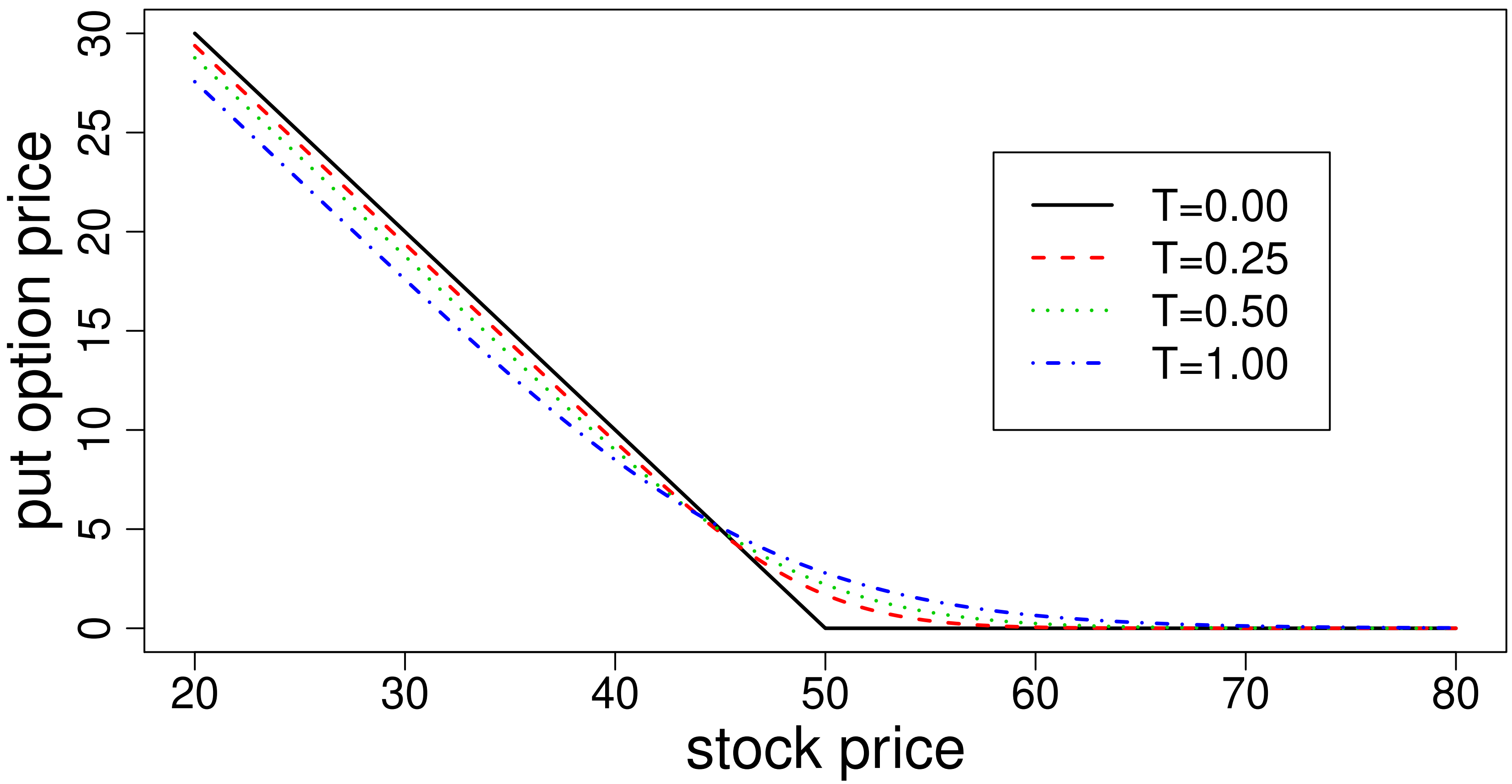
If the strike price S exceeds the stock price S_T at maturity, the option is *in the money*. In this case, it is optimal for the owner to exercise the option. The in-the-money option is worth the excess, if any, of the stock price and the strike price, which is the amount the owner pockets by exercising the option. *This assumes no frictions.*

If the stock price exceeds the strike price at maturity, the option is *out of the money* In this case, it is optimal for the owner to let the option expire without exercising it, because the exercise would cost the owner the difference between the stock price and the strike price. The price of the option is zero.

Summary: At maturity (T), the European put is worth $\max(0, X - S_T)$, which is the larger of zero and the net value of exercising.

An *American put option* is just like the European option, but the option of exercising is available anytime (once only) at or before maturity.

Put Price



Put price as a function of stock price and maturity

More on put options

The put option pays off when the underlying stock goes down but does not obligate the owner when the underlying stock goes up. For this privelege, the purchaser pays a price (*premium*) up front. The market price of the option depends on the exercise price, the stock price, the time to maturity, the volatility of the underlying stock, the riskless interest rate, and the anticipated size of dividends before maturity.

Same as for call options, the most important influence on the put's value day-to-day is changes in the underlying stock's price, and the next most important influence is changes in the volatility of the underlying stock. Put and call prices both increase when volatility increases, while they move in opposite directions (call up, put down) if the underlying stock price increases.

Sensitivities of options' market price

When this $\uparrow$	call price	put price
Stock price	$\uparrow$	$\downarrow$
Volatility	$\uparrow$	$\uparrow$
Strike price	$\downarrow$	$\uparrow$
Time to maturity	$\uparrow$	$\updownarrow$
Interest rate	$\uparrow$	$\downarrow$
Dividend	$\downarrow$	$\uparrow$

Since the largest changes on a day-to-day basis are the stock price and the volatility, we should think of buying a call as buying the stock and buying vol, and a call as selling the stock and buying vol.

In-class exercise: European put option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 5/4$ (interest rate is 25%). What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a European put option with a $\$125$ strike price maturing in two periods?

stock

European put

reminder: $\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}$.

Binomial model strategy: American versus European stock options

Binomial valuation follows the steps:

- Compute the tree from the underlying, starting at the beginning
- Evaluate the option at the end based on the contract terms
- Step back through the tree to compute the value one node at a time
  - For a European value is the value of holding into next period
  - For an American option, the value is the larger of holding it into the next period or exercising now.

In-class exercise: American put option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 5/4$ (interest rate is 25%). What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of an American put option with a $\$125$ strike price maturing in two periods?

stock

European put

reminder: $\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}$.

Futures are not assets!

Futures: $0 \quad \begin{array}{l} \nearrow F_u - F \\ \searrow F_d - F \end{array}$

$$0 = \pi_u^*(F_u - F) + \pi_d^*(F_d - F)$$

So $E^*[\Delta F] = 0$ or $F_t = E_t^*[F_{t+1}]$

Because we do not put money into a futures contract (and our margin account adjusted daily to leave no value in the futures), the price in risk-neutral probabilities has expected growth 0, not expected growth at the interest rate like assets.

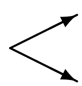
In-class exercise: Futures option valuation

Suppose the initial stock index is $\$50$ and that the stock price doubles or falls by half each period. Assume further that the riskfree rate is $5/4$. Assuming there are no dividends, compute the value to day of a call option maturing one period from now with strike $\$50$ on the index futures contract maturing two periods from now.

stock index

index futures

futures option



reminder: $\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}$.

Wrap-up

Binomial pricing of options!

- single-period: use risk-neutral probabilities and discounted expected value
- multiple-period: use single period valuation again and again
- strategy:
  - generate a valuation tree for the underlying
  - value the option at the terminal date based on contract terms
  - use single-period valuation to step through the tree
- call option example