DERIVATIVE SECURITIES
Lecture 3: Distribution-Free Results

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- what are distribution-free results?
- option pricing bounds
- early exercise
- spanning with options

Distribution-free results

So far, we have been using the binomial model to perform valuation. This model has strong explicit assumptions about underlying stock-price movements so we can compute the option prices exactly, or to be more precise we get a definite number that would be correct if our assumptions were true.[1] For the coming slides, we are going to consider *distribution-free* results that depend only on the absence of arbitrage or related concepts in a frictionless world and possibly positivity of the interest rate, and place little or no assumption on the distribution of the underlying.

[1] Of course, our assumptions are not exactly true. Useful models are always simpler than the real world.

Arbitrage and related concepts

Arbitrage is the most important concept in Finance. When we are looking for profit opportunities in trading or other financial decisions, we are usually best off thinking about that as a search for arbitrage opportunities. And, when we are valuing assets or positions, finding the valuation that does not lead to arbitrage is our most natural benchmark.

Definition An *arbitrage opportunity* is a strategy that never costs us anything now or in the future, in any contingency, but has a positive probability of having a positive cash flow at some date (or dates).

Definition The *law of one price* says that assets promising the same cash flows will have the same price.

Definition *Dominance*: One action dominates another if it leaves us better off some time in some contingency and never any worse off

An upper bound on the call option price

Throughout, assume there are no splits or assessments. There may be dividends except as noted. Suppose there is no arbitrage. Then the value of a European call option is not larger than the stock price.

Proof: Suppose not. Then consider the strategy of buying a share of stock, selling a call option, tendering the stock if the option is exercised, and selling the stock if the option expires unexercised. This strategy pays the difference in the prices up front and never costs anything. Furthermore, any dividends before exercise or maturity, the exercise price paid on exercise, and sale of stock when the option expires unexercised are all potential additional payoffs. Therefore, this strategy is an arbitrage opportunity, which is a contradiction.

An upper bound on the call option price: worksheet

Suppose that the call price now (C) is larger than the stock price now (S). Let S_T denote the stock price at maturity of the option.

time	now	maturity
write a call		
buy a call		
buy stock		
net cash flow		

A lower bound on the European call option price

Suppose there is no arbitrage and it is known that the stock will pay no dividends before the option's maturity. Then the value of a European call option is no smaller than the stock price less the present value of obtaining the exercise price at the option's maturity. Proof: Suppose not. Then consider the strategy of buying the option and *always*[2] exercising it, selling short the stock, buying a riskless bond that promises the exercise price at the option's maturity, and pocketing the remainder (which is positive by supposition). At maturity of the option, this is a wash, because the bond will pay for the exercise of the option and the share of stock received from exercise will undo the short position. Because there are no dividends, there are no other cash outflows associated with this strategy. Therefore the strategy is an arbitrage opportunity, which is a contradiction.

[2] not an optimal strategy, but good enough to run the arbitrage

A lower bound on the European call option price: worksheet

Suppose $C < S - Xe^{-r_f T}$.

time	now	maturity
buy a call		
sell stock		
lend money		
net cash flow		

In-class exercise: put option valuation bound

Supposing there is no arbitrage, show that the value of a European put option is no smaller than the present value of the strike price less the stock price. Assume a continuously compounded interest rate $r_f > 0$ so $\$1$ now grows to $\exp(r_f T)$ >1 at option maturity.

time	now	maturity
buy/sell a put		
buy/sell stock		
borrow/lend money		
net cash flow		

hint: set up the arb that would be present if the European put price P were smaller than $Xe^{-r_f T} - S$ where X is the strike price and S is the stock price.

European versus American call Options

Suppose investors use undominated strategies and the law of one price holds. Also assume it is known that a stock will pay no dividends before the option's maturity, and that the interest rate is positive. Then exercising an American call option on the stock before maturity is a dominated strategy, and therefore European and American call options have the same value.

Proof: Exercising an American call option now before maturity is dominated by pursuing the following alternative strategy. Today, obtain the same value by shorting the stock, pocketing the excess of the proceeds over the exercise price, and investing the exercise price in interest-bearing bonds. At the maturity of the option, exercise no matter what. The share of stock obtained from exercise offsets the short position. The bond principal covers the exercise price, and the interest is pocketed. Because this interest is the only diffence in payoff between the two strategies, the alternative strategy dominates exercising early. This means the two types of options have the same cash flows (since the exercise decision at maturity is the same), and therefore the law of one price implies the prices are the same.

Put-call parity theorem

Consider European put and call options on a a stock that pays no dividends. We assume that the options are matched in the sense that they have the same maturity date and the same exercise price. Consider also a riskless discount bond maturing at the same date as the options and having a face value equal to the common exercise price of the options.

The value of holding the stock and the put is the same as the value of holding the bond and the call, i.e., for all t,

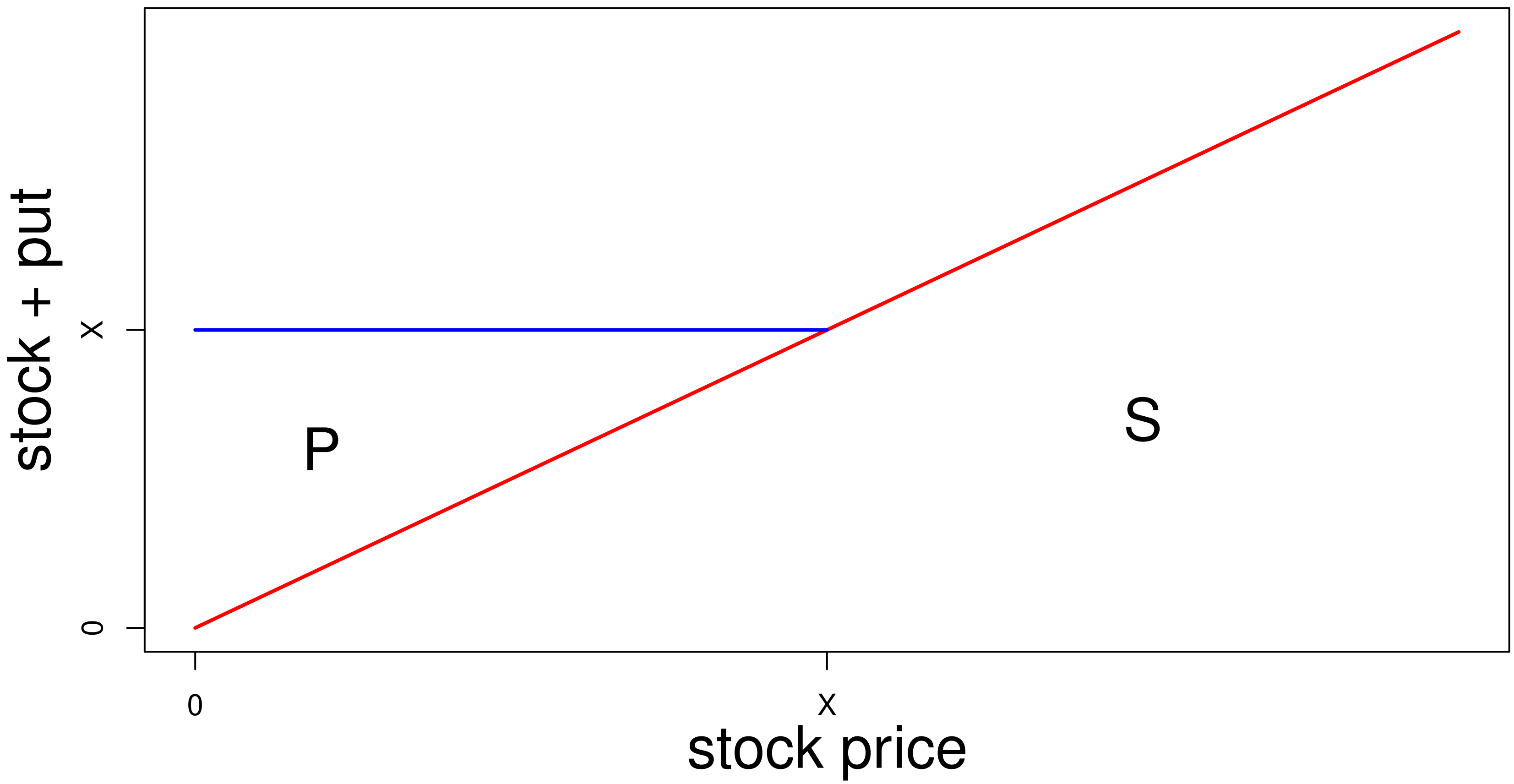
$$S_t + P_t = B_t + C_t.$$

Proof: Because we are working with European options, the value today depends only on the value at maturity. If $S_T \geq X$, $P_T = 0$, $C_T = S_T - X$, and $B_T = X$. Therefore, both $S_T + P_T$ and $B_T + C_T$ equal S_T.

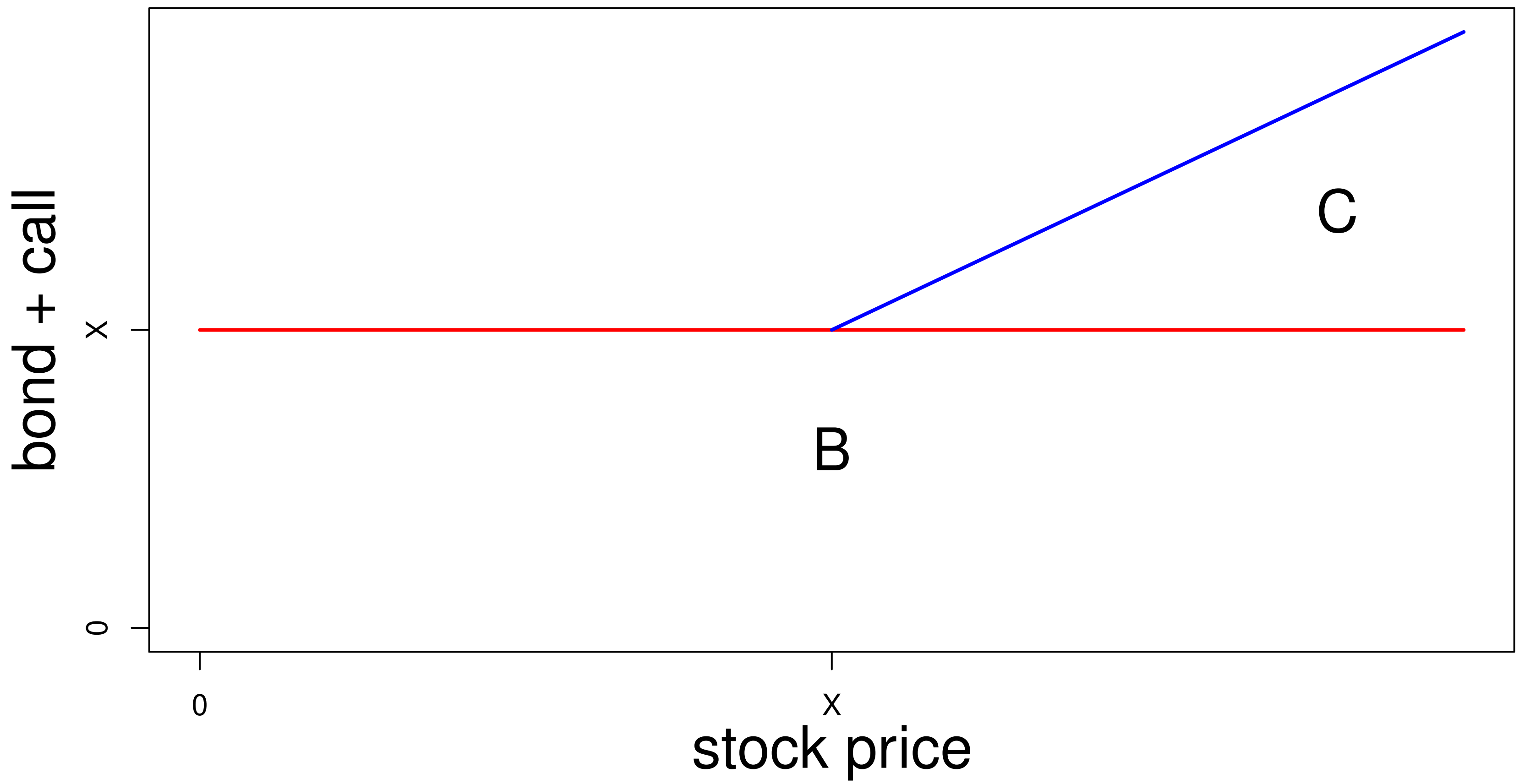
For $S_T < X$, $P_T = X - S_T$, $C_T = 0$, and $B_T = X$. Therefore, both $S_T + P_T$ and $B_T + C_T$ equal X.

Since $S_T + P_T = B_T + C_T$ in both cases, it must also be that $S_t + P_t = B_t + C_t$ for all $t < T$

Put-call parity: stock + put



Put-call parity: bond + call



In-class exercise: put-call parity

Hi-Tech Biceps is a start-up company selling electronic exercise gear. HB is not currently paying dividends, nor are dividends expected for the next two years. HB stock now sells for $\$55$ a share, and at-the-money HB European call options maturing one year from now are selling for $\$18$ apiece. The current riskless rate is 10%. (This is an ordinary interest rate, not a continuously compounded rate.) What is a fair price for at-the-money HB European put options maturing one year from now?

Spanning (replicating claims) using options

A very useful paper by Steve Ross points out that you can use options to replicate a very general function of the stock price:

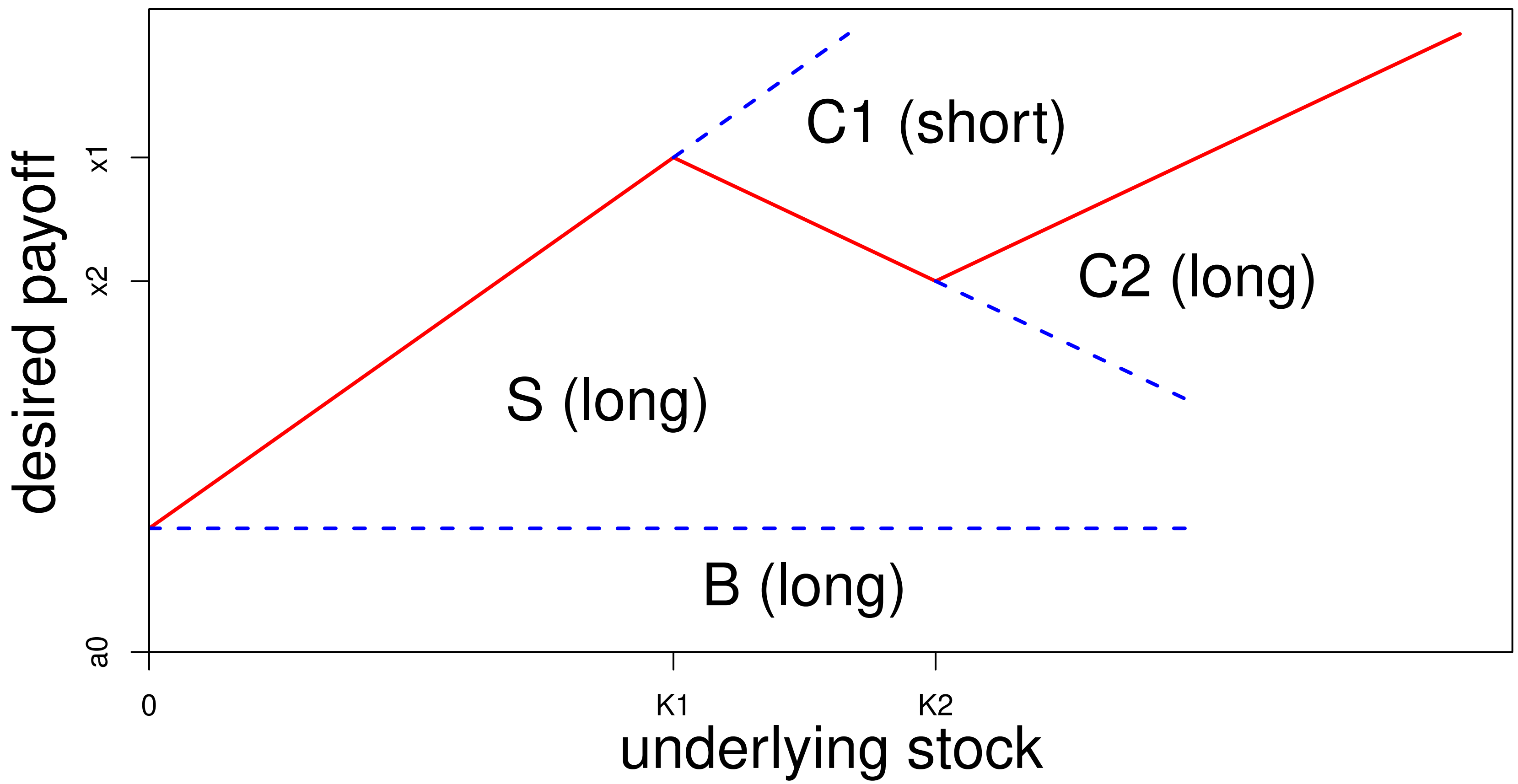
Ross, Stephen A., 1976, "Options and Efficiency," Quarterly Journal of Economics 90, pp 75–89.

One simple version of that result says we can replicate all continuous piecewise-linear claims using the stock, the bond, and call options with different strikes. Let the claim equal

$$H_T = \begin{cases} a_0 + b_0 S_T & 0 = K_0 \le S_T < K_1 \\ a_1 + b_1 S_T & K_1 \le S_T < K_2 \\ \vdots & \\ a_{n-1} + b_{n-1} S_T & K_{n-1} \le S_T < K_n \\ a_n + b_n S_T & K_n \le S_T \end{cases}$$

Here we want $a_i + b_i K_i = a_{i+1} + b_{i+1} K_i$ for all $i = 0, 1, ..., n-1$, which is the continuity requirement. How can we replicate this claim using traded call options, stock, and bonds?

Spanning a continuous piecewise-linear claim



Valuing the claim

In this case, we can write the option payoff at maturity as

$$H_T = a_0 B_T + b_0 S_T + \sum_{i=1}^{n} (b_i - b_{i-1}) C(S, K_i, T, T),$$

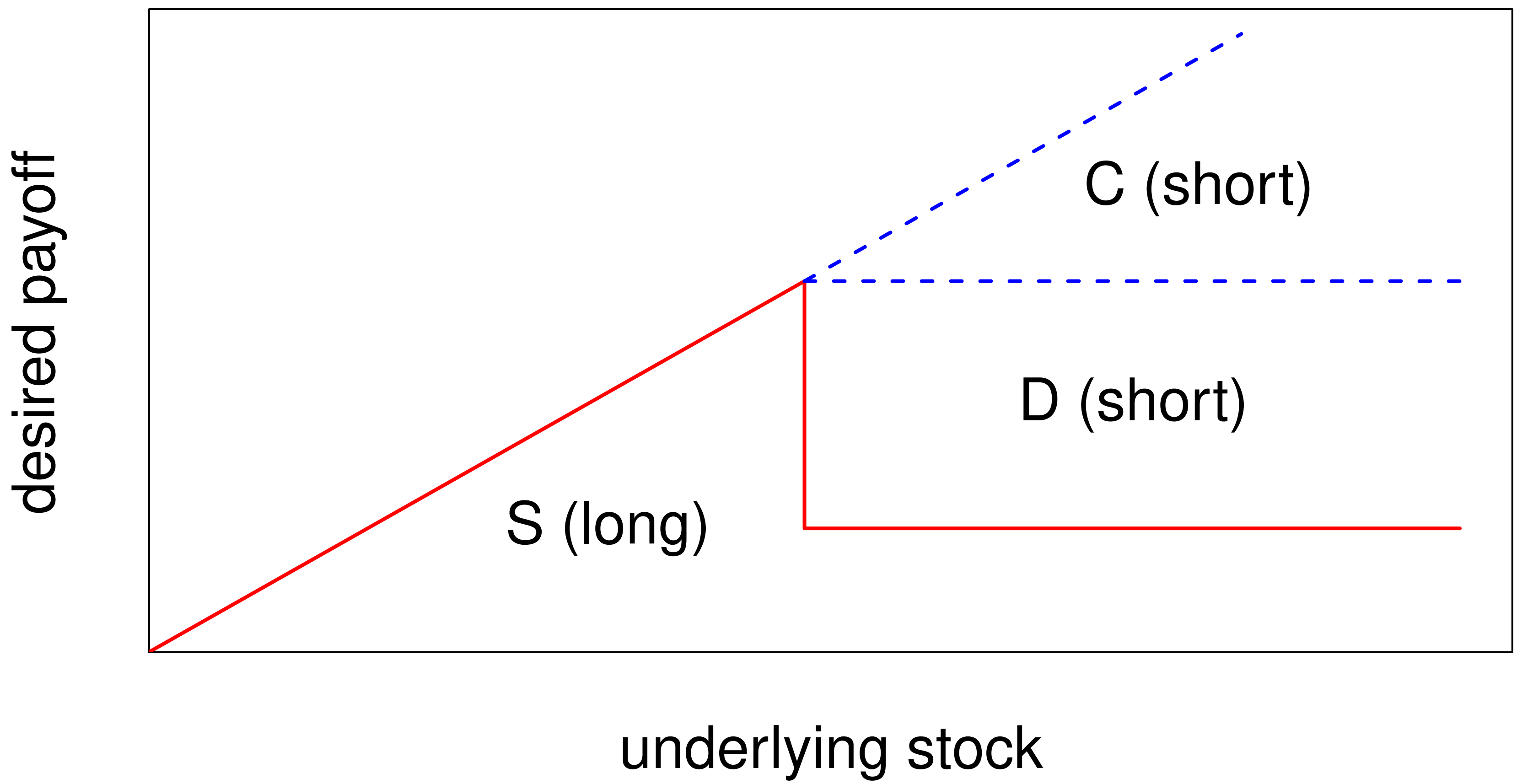
so we can replicate it by holding a_0 discount bonds maturing at T, b_0 net shares of stock ($=$ call options with strike 0), and $(b_i - b_{i-1})$ net long calls with strike K_i at each breakpoint i. The value of this position at an earlier time t is given by

$$H_t = a_0 B_t + b_0 S_t + \sum_{i=1}^{n} (b_i - b_{i-1}) C(S, K_i, t, T).$$

Spanning with options: further observations

- Spanning with options is very useful for planning trading programs or creating dynamic trading strategies like portfolio insurance.
- The spanning approach is more general and more useful than the old-fashioned approach used for talking about special strategies (bear spread, bull spread, butterfly spread, straddle, strangle etc.).
- These claims can also be replicated using put option (via put-call parity).
- In practice, include transaction costs in the payoffs and P&L of these strategies!
- Discontinuous claims can be spanned if we add digital options.
- Add "additional time" for smoothing (e.g. using Black-Scholes).

Spanning a discontinuous piecewise-linear claim



Spanning: smoothing using extra time