Problem Set 6: One-shot approach
FIN 539 Mathematical Finance
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1. State the Fundamental Theorem of Asset Pricing in words.

The following are equivalent:

(i) Absence of riskless arbitrage
(ii) Existence of a consistent positive linear pricing rule
(iii) Existence of a hypothetical agent who prefers more to less and has an optimal choice

2. Assume our standard continuous-time model with (1) a single risky asset with constant expected return μ and constant local standard deviation of returns σ, and (2) a riskfree asset with constant risk-free rate r. Recall that the state-price density is $\xi_t = \exp((-r - \kappa^2/2)t - \kappa Z_t)$, where $\kappa = (\mu - r)/\sigma$. Consider the one-shot choice problem for an agent with initial wealth w_0 and consumption only at the horizon T and utility function $u(c_T) = \log(c_T - \bar{c})$ where the constant $\bar{c}$ is the subsistence consumption.

A. Write down the choice problem.

Choose c_T to
maximize $\mathrm{E}[u(c_T)]$
s.t. $\mathrm{E}[\xi_T c_T] = w_0$

B. Write down the first-order condition and then write down optimal consumption as a function of ξ_T and the Lagrangian multiplier.

$u'(c_T) = \lambda \xi_T$

$\frac{1}{c_T - \bar{c}} = \lambda \exp((-r - \kappa^2/2)T - \kappa Z_T)$

$c_T = \bar{c} + \frac{1}{\lambda \exp((-r-\kappa^2/2)T - \kappa Z_T)}$

C. Solve for the wealth process w_t in terms of λ and Z_t. (Recall that for

$t > s$, $Z_t - Z_s$ is independent of the history and distributed normally with mean 0 and variance $t-s$. Also, recall that if X is distributed normally with mean a and variance b then $E[\exp(X)] = \exp(a + b/2)$.)

The valuation equation says that

$$\begin{aligned}
w_t &= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} c_T\right] \\
&= \mathrm{E}_t\left[\frac{\exp((-r-\kappa^2/2)T - \kappa Z_T)}{\exp((-r-\kappa^2/2)t - \kappa Z_t)}\left(\bar{c} + \frac{1}{\lambda \exp((-r-\kappa^2/2)T - \kappa Z_T)}\right)\right] \\
&= \bar{c} E_t\left[\exp((-r-\kappa^2/2)(T-t) - \kappa(Z_T - Z_t))\right] + \frac{1}{\lambda}\exp((r+\kappa^2/2)t + \kappa Z_t) \\
&= \bar{c}\exp(-r(T-t)) + \frac{1}{\lambda}\exp((r+\kappa^2/2)t + \kappa Z_t)
\end{aligned}$$

D. Solve for λ, and use this value to restate the wealth process.

From the budget constraint and the result in C, we have that $w_0 = \bar{c}e^{-rT} + 1/\lambda$, and therefore

$$\lambda = \frac{1}{w_0 - \bar{c}e^{-rT}}.$$

Substituting into the wealth process, we have

$$w_t = \bar{c}\exp(-r(T-t)) + (w_0 - \bar{c}e^{-rT})\exp((r+\kappa^2/2)t + \kappa Z_t)$$

E. Solve for the portfolio allocation for the risky asset as a function of wealth and time.

Given there is no intermediate consumption, we can write the usual budget constraint as

$$dw_t = r w_t dt + \theta((\mu - r)dt + \sigma dZ_t)$$

and from our expression for wealth, we can compute

$$dw_t = (...)dt + (w_0 - \bar{c}e^{-rT})\exp((r + \kappa^2/2)t + \kappa Z_t)\kappa dZ_t$$

(where the term (...) terms from the drift of w, direct dependence on t, and the Itô term)

Matching coefficients on dZ_t and then substituting in for w_t, we have that

$$\begin{aligned}\theta &= (w_0 - \bar{c}e^{-rT})\exp((r + \kappa^2/2)t + \kappa Z_t)\kappa/\sigma \\ &= (w_t - \bar{c}\exp(-r(T-t)))\kappa/\sigma.\end{aligned}$$

In this expression, $\bar{c}\exp(-r(T-t))$ is the amount of wealth at t needed to fund the minimum consumption $\bar{c}$, so the factor $(w_t - \bar{c}\exp(-r(T-t)))$ is the discretionary wealth beyond what is needed to meet minimum consumption.