Mathematical Finance Mini Exam, Spring B 2021
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This is a closed-book exam: you may not use any books, notes, or electronic devices (calculators, headphones, laptops, etc.), except for the Zoom session for proctoring (camera must be on), reading the exam, asking me questions, and for submitting your exam. Mark your answers on paper and submit pictures of your answer sheets in Canvas.

There are no trick questions on the exam, but you should read the questions carefully.

PLEDGE (required)

The work on this exam will be mine alone, and I will conform with the rules of the exam and the code of conduct of the Olin Business School.

Signed name ______________________________

Good luck!

I. Short answer (30 points).

A. State in words the Fundamental Theorem of Asset Pricing (FTAP).

B. What is the difference between priced risk in the Intertemporal Capital Asset Pricing Model (ICAPM) and the APT (Arbitrage Pricing Theory)?

C. Why is it reasonable to assume that utility functions are strictly concave, and why is this useful?

II. Bellman equation (40 points) Consider a continuous-time portfolio choice problem with power felicity function $u(c) = c^{1-R}/(1-R)$ for consumption over an infinite horizon with pure rate of time discount ρ. There is a constant riskfree rate $r > 0$ and a single risky asset with constant expected return $\mu > r$ per unit time and constant local variance σ^2 per unit time. The choice problem is

Given w_0 at time 0,
choose adapted θ_s and w_s to
maximize $\mathrm{E}[\int_{s=0}^{\infty} e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds]$
s.t. $(\forall s)(dw_s = rw_s ds + \theta_s((\mu - r)ds + \sigma dZ_s) - c_s ds)$
$(\forall s)(w_s \geq 0)$

A. Write down the process M_t for this problem.

B. What does M_t represent given the optimal policies for the portfolio and wealth?

What does M_t represent given an arbitrary policy?

For $t > s$, what is $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

C. Derive the Bellman equation for this problem.

D. Solve for optimal c_t and θ_t in terms of derivatives of V.

E. It is possible to exploit homogeneity to prove that the value function is of the form $V(w) = vW^{1-R}/(1-R)$ for some constant v. Use this representation (don't prove!) to solve for optimal consumption and portfolio as a function of wealth and v.

III. One-shot approach (30 points) Assume our standard continuous-time model with (1) a single risky asset with constant expected return μ and constant local standard deviation of returns σ, and (2) a riskfree asset with constant risk-free rate r. Recall that the state-price density is $\xi_t = \exp((-r-\kappa^2/2)t - \kappa Z_t)$, where $\kappa = (\mu - r)/\sigma$. Consider the one-shot choice problem for an agent with initial wealth W_0 and consumption over time with an infinite horizon, felicity function $u(c_t) = \log(c_t)$, and pure rate of time discount ρ.

A. Write down the one-shot choice problem.

B. Write down the first-order condition for optimal consumption and then write optimal consumption as a function of the state-price density and the Lagrangian multiplier.

C. What is the equation we would solve for λ? Solve for λ.

D. What is the equation we would solve for the wealth process w_t? (You need not solve it.)

IV. Challenger (10 bonus points) Consider the standard infinite-horizon problem with fixed coefficients and constant relative risk aversion R in II above.

For what values of the parameters μ, $\sigma > 0$, r, $R \in (0, 1)$, and $\rho > 0$ does the problem have a bounded value? Explain the economics of your result; prove your claim for full credit. (This is hard: don't work on this problem until you have completed and checked everything else.)

Some formulas that might be useful

univariate Itô's lemma:

Let $dX_t = a_t dt + \sigma_t dZ_t$ where Z is a standard Wiener process, and let $f(X,t)$ have continuous partial derivatives f_X, f_{XX}, and f_t. Then

$$df(X_t,t) = f_X(X_t,t)(a_t dt + \sigma_t dZ_t) + f_t(X_t,t)dt + \frac{\sigma_t^2}{2} f_{XX}(X_t,t)dt.$$

multivariate Itô's lemma:

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where Z_t is an m-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY = H_t dt + H_x dX + \frac{1}{2}\text{tr}(GG' H_{xx})dt$$

Note: if H takes values in $\Re^K$, we can apply the result elementwise.

Black-Scholes differential equation:

$$0 = -r\mathcal{O} + \mathcal{O}_t + rS\mathcal{O}_S + \frac{\sigma^2}{2}S^2\mathcal{O}_{SS},$$

State-price density (stochastic discount factor) if markets are complete:

Let security 0 have a riskless mean return r and any other asset $n = 1, ..., N$ has re-invested risky return $dS_{nt}/S_{nt} = \mu_{nt} t dt + \gamma_{nt} dZ_t$.

$$d\xi = -rdt - (\mu - r\mathbf{1})'(\Gamma')^{-1}dZ_t$$

where

$$\Gamma = (\gamma_1 | \gamma_2 | \ldots | \gamma_N)'.$$

Univariate state-price density:

$$d\xi_t/\xi_t = -rdt - \kappa dZ_t,$$

where $\kappa \equiv (\mu - r)/\sigma$, and with constant coefficients and taking $\xi_0 = 1$ wlog, we have

$$\xi_t = \xi_0 \exp((-r - \kappa^2/2)t - \kappa Z_t),$$

Normal moment generating function:

If $x \sim N(m, s)$, $\mathrm{E}[e^x] = e^{m+s^2/2}$

Arrow-Pratt coefficient of absolute risk aversion:

$\frac{-u''(c)}{u'(c)}$

Arrow-Pratt coefficient of relative risk aversion:

$\frac{-cu''(c)}{u'(c)}$

Constant Absolute Risk Aversion (CARA) utility with risk aversion $A > 0$:

$$u(c) = -\frac{\exp(-Ac)}{A}$$

Constant Relative Risk Aversion (CRRA) utility with risk aversion $R > 0$:

$$u(c) = \begin{cases} \frac{c^{1-R}}{1-R} & \text{for } R \neq 1 \\ \log(c) & \text{for } R = 1 \end{cases}$$

Kuhn-Tucker conditions:

For the optimization model

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$,

the Kuhn-Tucker conditions are

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$