Mathematical Finance Mini Exam Answers, Spring A 2024
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This is a closed-book exam: you may not use any books, notes, or electronic devices (calculators, headphones, cell phones, laptops, etc.) during the exam. Mark your answers on the provided blue books. Make sure each answer is clearly indicated. BE SURE TO PUT YOUR NAME ON THE BLUE BOOK! There are no trick questions on the exam, but you should read the questions carefully.

PLEDGE (required)

The work on this exam will be mine alone, and I will conform with the rules of the exam and the Code of Conduct of the Olin Business School.

Signed name ______________________________

Good luck!

I. Short answer (30 points).

A. In words, what does it mean when an agent has constant absolute risk aversion?

The preferences for gambles of the same absolute size are the same at all wealth levels.

B. What is the difference between priced risk in the Capital Asset Pricing Model (CAPM) and the Consumption CAPM (CCAPM)?

In the CAPM, the priced risk is the market portfolio, but in the CCAPM the priced risk is changes in consumption.

C. If a client gives you a covariance matrix of returns to use, why should you look at its eigenvalues? What are you looking for? What should you do if there is a problem?

If any eigenvalue is negative, then it is not a valid covariance matrix because some portfolios have negative variance. This causes problems with portfolio optimization and makes it impossible to simulate returns. The problem can be fixed by replacing any negative eigenvalues with small positive ones.

II. Bellman equation (40 points) Consider a continuous-time portfolio choice problem with power felicity function $u(c) = c^{1-R}/(1-R)$ for consumption over an infinite horizon with pure rate of time discount ρ. There is a constant riskfree rate $r > 0$ and a single risky asset with constant expected return $\mu > r$ per unit time and constant local variance σ^2 per unit time. The choice problem is

Given w_0 at time 0,
choose adapted θ_s and w_s to
maximize $\mathrm{E}[\int_{s=0}^{\infty} e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds]$
s.t. $(\forall s)(dw_s = rw_s ds + \theta_s((\mu - r)ds + \sigma dZ_s) - c_s ds)$
$(\forall s)(w_s \geq 0)$

A. Write down the process M_t for this problem.

$$M_t = \int_{s=0}^{t} e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds + e^{-\rho t} V(w_t, t)$$

B. What does M_t represent given the optimal policies for the portfolio and wealth?

the conditional expected utility of the optimal strategy as of time t

What does M_t represent given an arbitrary policy?

the conditional expected utility at time t of following the arbitrary policy until time t and the optimal policy from then on

For $t > s$, what is $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

the loss in value of the objective function due to mistakes made between time s and time t

C. Derive the Bellman equation for this problem.

$$\begin{aligned}\frac{\mathrm{E}[dM_t]}{e^{-\rho t}dt} &= \frac{c^{1-R}}{1-R} - \rho V + V_t + V_w\frac{E[dw]}{dt} + \frac{1}{2}\frac{E[dw^2]}{dt}V_{ww} \\ &= \frac{c^{1-R}}{1-R} - \rho V + V_t + (rw + \theta(\mu - r) - c)V_w + \frac{\theta^2\sigma^2}{2}V_{ww}\end{aligned}$$

The Bellman equation says that the maximum over the controls of $\mathrm{E}[dM_t]$ equals zero, so the Bellman equation is

$$0 = \max_{\theta,c}\left(\frac{c^{1-R}}{1-R} - \rho V + V_t + (rw + \theta(\mu - r) - c)V_w + \frac{\theta^2\sigma^2}{2}V_{ww}\right)$$

D. Solve for optimal c_t and θ_t in terms of derivatives of V.

The terms that depend on c are $c^{1-R}/(1-R)$ and $-cV_w$. Differentiating with respect to c, we have the first-order condition

$$c^{-R} - V_w = 0,$$

and therefore

$$c^* = (V_w)^{-1/R}.$$

The terms that depend on θ are $\theta(\mu - r)V_w$ and $\theta^2\sigma^2 V_{ww}/2$. Differentiating with respect to θ, we have the first-order condition

$$(\mu - r)V_w + \theta\sigma^2 V_{ww} = 0,$$

which implies the optimal portfolio is

$$\theta^* = \frac{\mu - r}{\sigma^2}\frac{1}{-V_{ww}/V_w}.$$

E. It is possible to exploit homogeneity to prove that the value function is of the form $V(w) = vW^{1-R}/(1-R)$ for some constant v. Use this representation (don't prove!) to solve for optimal consumption and portfolio as a function of wealth and v.

Given the form of $V(w_t, t)$, we have that

$$
\begin{aligned}
V &= vw^{1-R}/(1-R) \\
V_w &= vw^{-R} \\
V_{ww} &= -Rvw^{-R-1} \\
-V_{ww}/V_w &= R/w
\end{aligned}
$$

Therefore,

$$c^* = (V_w)^{-1/R} = (vw^{-R})^{-1/R} = v^{-1/R}w$$

and

$$
\begin{aligned}
\theta^* &= \frac{\mu - r}{\sigma^2}\frac{1}{R/w} \\
&= \frac{\mu - r}{\sigma^2 R}w
\end{aligned}
$$

III. One-shot approach (30 points) Assume our standard continuous-time model with (1) a single risky asset with constant expected return μ and constant local standard deviation of returns σ, and (2) a riskfree asset with constant risk-free rate r. Recall that the state-price density is $\xi_t = \exp((-r-\kappa^2/2)t - \kappa Z_t)$, where $\kappa = (\mu - r)/\sigma$. Consider the one-shot choice problem for an agent with initial wealth W_0 and consumption over time with an infinite horizon, felicity function $u(c_t) = \log(c_t)$, and pure rate of time discount ρ.

A. Write down the one-shot choice problem.

Given w_0 at time 0,
choose adapted c_t to

maximize $\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t}\log(c_t)dt]$
st $\mathrm{E}[\int_{t=0}^{\infty} \exp((-r-\kappa^2/2)t-\kappa Z_t)c_t] = w_0$

B. Write down the first-order condition for optimal consumption and solve for optimal consumption as a function of the state-price density and the Lagrangian multiplier.

$$e^{-\rho t}\frac{1}{c_t} = \lambda \exp((-r-\kappa^2/2)t-\kappa Z_t)$$

$$c_t^* = \frac{1}{\lambda}\exp((r-\rho+\kappa^2/2)t+\kappa Z_t)$$

C. What is the equation we would solve for λ? Solve for λ.

the budget constraint

$$\begin{aligned} w_0 &= \mathrm{E}[\int_{t=0}^{\infty} \exp((-r-\kappa^2/2)t-\kappa Z_t)c_t dt] \\ &= \mathrm{E}[\int_{t=0}^{\infty} \exp((-r-\kappa^2/2)t-\kappa Z_t)\frac{1}{\lambda}\exp((r-\rho+\kappa^2/2)t+\kappa Z_t)dt] \\ &= \mathrm{E}[\int_{t=0}^{\infty} \frac{1}{\lambda}\exp(-\rho t)dt] \\ &= \frac{1}{\lambda\rho} \end{aligned}$$

so $\lambda = 1/(w_0\rho)$.

D. What is the equation we would solve for the wealth process w_t? (You need not solve it.)

$$w_t = \mathrm{E}_t\left[\int_{s=t}^{\infty} \frac{\xi_s}{\xi_t} c_s ds\right]$$

IV. Challenger (10 bonus points) Consider the standard infinite-horizon problem with fixed coefficients and constant relative risk aversion R in II above.

For what values of the parameters μ, $\sigma > 0$, r, $R \in (0, 1)$, and $\rho > 0$ does the problem have a bounded value? Explain the economics of your result; prove your claim for full credit. (This is hard: don't work on this problem until you have completed and checked everything else.)

Some formulas that might be useful

univariate Itô's lemma:

Let $dX_t = a_t dt + \sigma_t dZ_t$ where Z is a standard Wiener process, and let $f(X,t)$ have continuous partial derivatives f_X, f_{XX}, and f_t. Then

$$df(X_t,t) = f_X(X_t,t)(a_t dt + \sigma_t dZ_t) + f_t(X_t,t)dt + \frac{\sigma_t^2}{2} f_{XX}(X_t,t)dt.$$

multivariate Itô's lemma:

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where Z_t is an m-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY = H_t dt + H_x dX + \frac{1}{2}\mathrm{tr}(GG' H_{xx})dt$$

Note: if H takes values in $\Re^K$, we can apply the result elementwise.

Black-Scholes differential equation:

$$0 = -r\mathcal{O} + \mathcal{O}_t + rS\mathcal{O}_S + \frac{\sigma^2}{2}S^2\mathcal{O}_{SS},$$

State-price density (stochastic discount factor) if markets are complete:

Let security 0 have a riskless mean return r and any other asset $n = 1, ..., N$ has re-invested risky return $dS_{nt}/S_{nt} = \mu_{nt} t dt + \gamma_{nt} dZ_t$.

$$d\xi = -rdt - (\mu - r\mathbf{1})'(\Gamma')^{-1} dZ_t$$

where

$$\Gamma = (\gamma_1 | \gamma_2 | \dots | \gamma_N)'.$$

Univariate state-price density:

$$d\xi_t/\xi_t = -rdt - \kappa dZ_t,$$

where $\kappa \equiv (\mu - r)/\sigma$, and with constant coefficients and taking $\xi_0 = 1$ wlog, we have

$$\xi_t = \xi_0 \exp((-r - \kappa^2/2)t - \kappa Z_t),$$

Normal moment generating function:

If $x \sim N(m, s)$, $\mathrm{E}[e^x] = e^{m+s^2/2}$

Arrow-Pratt coefficient of absolute risk aversion:

$\frac{-u''(c)}{u'(c)}$

Arrow-Pratt coefficient of relative risk aversion:

$\frac{-cu''(c)}{u'(c)}$

Constant Absolute Risk Aversion (CARA) utility with risk aversion $A > 0$:

$$u(c) = -\frac{\exp(-Ac)}{A}$$

Constant Relative Risk Aversion (CRRA) utility with risk aversion $R > 0$:

$$u(c) = \begin{cases} \frac{c^{1-R}}{1-R} & \text{for } R \neq 1 \\ \log(c) & \text{for } R = 1 \end{cases}$$

Kuhn-Tucker conditions:

For the optimization model

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$,

the Kuhn-Tucker conditions are

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$