Mathematical Finance Mini Exam Answers, Spring A 2025
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I. Short answer (30 points).

A. In words, what does it mean when one agent is more risk averse than another?

Given a choice between a sure thing and a risky gamble, the more risk averse agent will choose the sure thing in strictly more situations than the less risk averse agent.

B. What is the difference between priced risk in the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT)?

In the CAPM, market risk is the only priced risk, but in the APT any common risk in the cross-section of stocks can be priced.

C. If a client gives you a covariance matrix of returns to use, what should you check? What should you do if there is a problem?

You should check to make sure the matrix is at least positive semi-definite (all eigenvalues nonnegative), or better yet positive definite (all eigenvalues positive). If not at least semi-definite, it is not a valid covariance matrix, causing simulation to fail and making optimization results meaningless.

II. Bellman equation (40 points) Consider a continuous-time portfolio choice problem with felicity function $u(c) = \log(c - \bar{c})$ for consumption over an infinite horizon with pure rate of time discount ρ. There is a constant riskfree rate $r > 0$ and a single risky asset with constant expected return $\mu > r > 0$ per unit time and constant local variance $\sigma^2 > 0$ per unit time. The choice problem is

Given $w_0 > 0$,
choose adapted $\{\theta_s\}$, $\{c_s\}$, and $\{w_s\}$ to
maximize $\mathrm{E}[\int_{s=0}^{\infty} e^{-\rho s} \log(c_s - \bar{c})]$
s.t. $(\forall s)(dw_s = rw_s ds + \theta_s((\mu - r)ds + \sigma dZ_s) - c_s ds)$
$(\forall s)(w_s \geq 0)$

A. Write down the process M_t for this problem.

$$M_t = \int_{s=0}^{t} e^{-\rho s} \log(c_s - \bar{c}) ds + e^{-\rho t} V(w_t)$$

B. What does M_t represent given the optimal policies for the portfolio and wealth?

M_t is the conditional expected value at t of the objective function given that we follow the optimal strategy for all time.

What does M_t represent given an arbitrary candidate policy?

M_t is the conditional expected value at t of the objective function given that we follow the candidate strategy until t and the optimal strategy from then on.

For $t > s$, how can we interpret $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

$\mathrm{E}[M_s] - \mathrm{E}[M_t]$ is the cost in units of the objective function of mistakes made between s and t.

C. Derive the Bellman equation for this problem.

$$0 = \max_{c,\theta} \frac{E[dM]}{e^{-\rho t} dt} \tag{1}$$

$$= \tag{2}$$

$$\max_{c,\theta} (\log(c_t - \bar{c}) - \rho V + (rw + \theta(\mu - r) - c))V_w + \frac{\theta^2 \sigma^2}{2} V_{ww}) \tag{3}$$

(For partial credit, it is a good idea to show some intermediate steps.)

D. Solve for optimal c and θ in terms of derivatives of V.

$u'(c) = V_w$, i.e., $1/(c - \bar{c}) = V_w$, so $c = \bar{c} + 1/V_w$.

$(\mu - r)V_w + \theta \sigma^2 V_{ww} = 0$, so $\theta = ((\mu - r)/\sigma^2)/(-V_{ww}/V_w).)$

E. It is possible to exploit scaling to prove that the value function is of the form $V(w) = \log(w_0 - \bar{c}/r)/\rho + v$ for some constant v. (Take it as given.)

Use this representation to solve for optimal consumption and portfolio as a function of wealth and v.

First-order condition for consumption: $u'(c) = V_w$. Substituting the form of $u(c)$ and $V(w)$, we have that $1/(c - \bar{c}) = 1/((w - \bar{c}/r)\rho)$. Solving for c, we have $c = \bar{c} + \rho(w - \bar{c}/r)$.

First-order condition for portfolio choice: $(\mu - r)V_w + \theta\sigma^2 V_{ww} = 0$. Substituting the form of $V(\cdot)$, we have that $(\mu - r)/((w - \bar{c}/r)\rho) - \theta\sigma^2/((w - \bar{c}/r)^2\rho)$. Solving for θ, we have $theta = ((\mu - r)/\sigma^2)(w - \bar{c}/r)$.

III. Homotheticity (30 points) Consider an infinite-horizon continuous-time portfolio choice problem with power felicity $u(c) = c^{1-R}/(1-R)$ for consumption, where $R > 0$ and $R \neq 1$, and a pure rate of time discount $\rho > 0$. There is a constant riskfree rate $r > 0$ and a single risky asset with expected return $\mu > r$ per unit time and local variance $\sigma^2 > 0$ per unit time. Then the choice problem is

Given $w_0 > 0$,
choose adapted θ_t, c_t, and w_t to
maximize $\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t}\frac{c_t^{1-R}}{1-R}dt]$
s.t. $(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt)$
$(\forall t)(w_t \geq 0)$

We want to show that the value function has a special form.

A. Rewrite the choice problem in terms of normalized variables.

Let $\hat{\theta}_t \equiv \theta_t/w_0$, $\hat{c}_t \equiv c_t/w_0$, and $\hat{w}_t \equiv w_t/w_0$. Then the choice problem can be rewritten as:

Given $w_0 > 0$,
choose adapted $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ to
maximize $\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t}\frac{(w_0\hat{c}_t)^{1-R}}{1-R}dt] = w_0^{1-R}\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t}\frac{\hat{c}_t^{1-R}}{1-R}dt]$
s.t. $(\forall t)(d\hat{w}_t = r\hat{w}_t dt + \hat{\theta}_t((\mu - r)dt + \sigma dZ_t)) - \hat{c}_t dt$
$(\forall t)(\hat{w}_t \geq 0)$

B. Note that the optimal solution in the normalized variables is the same independent of w_0 (explain why), and use this result to show that the value function can be written in the form $V(w_0) = v{w_0}^{1-R}$.

The only place w_0 appears in the rewritten choice problem is in the positive factor ${w_0}^{1-R}$ multiplying the objective function. In particular, the constraints and the factor multiplying ${w_0}^{1-R}$ in the objective function depend only on the transformed variables. The solution is the same for all w_0, because maximizing a positive constant times the objective function also maximizes the objective function. Let v be the value of the optimal solution when $w_0 = 1$, the objective function for arbitrary $w_0 > 0$ is multiplied by the constant ${w_0}^{1-R}$, so the value function is $V(w_0) = v{w_0}^{1-R}$

C. Would the result in part B still be true (perhaps for a different constant v) if we added the constraint $(\forall t)c_t \geq 5$? Explain why it is still true or why the proof fails.

It would not be true because the constraint depends on w_0 and not just the normalized variables. In particular, if we divide both sides by w_0, the constraint becomes $(\forall t)\hat{c}_t \geq 5/w_0$, and the set of available values for $\hat{c}_t$ depends on w_0, and the optimum without this constraint is not feasible (let alone optimal) for all w_0, t, and state of nature.