Problem Set 1: Kuhn-Tucker conditions
FIN 539 Mathematical Finance
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1. A complete-markets portfolio choice problem.

Given initial wealth w_0,
choose state-contingent consumptions c_1, ... c_Ω, to
maximize $\mathrm{E}[u(c_\omega)]$ (objective function)
st $\mathrm{E}[\xi_\omega c_\omega] = w_0$ (budget constraint)
$(\forall\omega) c_\omega \geq \bar{c}$ (consumption floor).

In this problem, $\mathrm{E}[u(c)]$ is the von Neumann-Morgenstern utility function with $u'(c) > 0$ and $u''(c) < 0$, $\omega = 1, ..., \Omega$ are the states of nature, ξ is the vector of state-price densities (or stochastic discount factors), and $\bar{c}$ is an exogenously-imposed floor on consumption.

A. What are the Kuhn-Tucker (KT) conditions for the problem?

The equality constraint can be written as $g(c) = 0$ where $g(c) = \mathrm{E}[\xi_\omega c_\omega] - w_0$, and the inequality constraints can be written as $g_\omega(c) \leq 0$, where $g_\omega(c) = \bar{c} - c_\omega$. Therefore, the KT conditions are

$$u'(c_\omega) = \lambda\xi_\omega - \lambda_\omega$$

$$(\forall\omega)\lambda_\omega \geq 0$$

$$(\forall\omega)(\bar{c} - c_\omega)\lambda_\omega = 0$$

B. Show that the solution to the Kuhn-Tucker conditions is given by

$$c_\omega = \max(I(\lambda\xi_\omega), \bar{c}),$$

where $I(z)$ is the inverse function of the marginal utility $u'(c)$ and λ is the Lagrangian multiplier on the budget constraint.

First, consider states ω in which $I(\lambda\xi_\omega) < \bar{c}$. In these states, we must have $\lambda_\omega \neq 0$, since $\lambda_\omega = 0$ would imply $c_\omega = I(\lambda\xi_\omega) < \bar{c}$, which is infeasible. By complementarity slackness $(\bar{c} - c_\omega)\lambda_\omega = 0$, $\lambda_\omega \neq 0$ implies that $c_\omega = \bar{c}$, so we have that $I(\lambda\xi_\omega) < \bar{c}$ implies that $c_\omega = \bar{c}$.

Next, consider states ω in which $I(\lambda\xi_\omega) \geq \bar{c}$. We show that in these states, $\lambda_\omega = 0$. Suppose not. By the KT conditions, $\lambda_\omega \geq 0$ so we have $\lambda_\omega > 0$. Since $u'(c_\omega) = \lambda\xi_\omega - \lambda_\omega$, $c_\omega = I(\lambda\xi_\omega - \lambda_\omega) > I(\lambda\xi_\omega) \geq \bar{c}$ (where we used $\lambda > 0$, $I(\cdot)$ decreasing because $u'(\cdot)$ decreasing, and the maintained assumption in this paragraph that $I(\lambda\xi_\omega) \geq \bar{c}$). However, $\lambda_\omega > 0$ and complementarity slackness $(\bar{c} - c_\omega)\lambda_\omega = 0$ imply that $c_\omega = \bar{c}$. Therefore, we have shown that if $I(\lambda\xi_\omega) \geq \bar{c}$, then $c_\omega = I(\lambda\xi_\omega)$.

Combining the two results, we have that

$$
\begin{aligned}
c_\omega &= \begin{cases} \bar{c} & \text{if } I(\lambda\xi_\omega) \leq \bar{c} \\ I(\lambda\xi_\omega) & \text{otherwise} \end{cases} \\
&= \max(\bar{c}, I(\lambda\xi_\omega))
\end{aligned}
$$

C. Suppose that $u(c) = \sqrt{k_1^2 + 4k_2c} + k_1 \log(\sqrt{k_1^2 + 4k_2c} - k_1)$; this is a special case of GOBI preferences.[1] Then show that

$$
I(z) = \frac{k_1}{z} + \frac{k_2}{z^2}.
$$

$$
\begin{aligned}
z &= u'(c) \\
&= \frac{2k_2}{\sqrt{k_1^2 + 4k_2c}} \left(1 + \frac{k_1}{\sqrt{k_1^2 + 4k_2c} - k_1}\right) \\
&= \frac{2k_2}{\sqrt{k_1^2 + 4k_2c}} \left(\frac{\sqrt{k_1^2 + 4k_2c} - k_1 + k_1}{\sqrt{k_1^2 + 4k_2c} - k_1}\right)
\end{aligned}
$$

[1] Dybvig, Philip H., and Fang Liu, 2018, On Investor Preferences and Mutual Fund Separation, Journal of Economic Theory 174, 224–260.

$$= \frac{2k_2}{\sqrt{k_1^2 + 4k_2 c} - k_1}$$

Therefore,

$$\sqrt{k_1^2 + 4k_2 c} - k_1 = \frac{2k_2}{z}$$

$$\sqrt{k_1^2 + 4k_2 c} = \frac{2k_2}{z} + k_1$$

$$k_1^2 + 4k_2 c = \frac{4k_2^2}{z^2} + \frac{4k_1 k_2}{z} + k_1^2$$

$$c = \frac{k_1}{z} + \frac{k_2}{z^2}$$

D. Given the choice of the utility in part C, write down the form of consumption.

$$c_\omega = \max\left(\bar{c}, \frac{k_1}{\lambda \xi_\omega} + \frac{k_2}{(\lambda \xi_\omega)^2}\right)$$

E. Write down the equation that should be solved for the Lagrange multiplier.

The budget constraint:

$$\mathrm{E}\left[\xi_\omega \max\left(\bar{c}, \frac{k_1}{\lambda \xi_\omega} + \frac{k_2}{(\lambda \xi_\omega)^2}\right)\right] = w_0$$