Mathematical Finance Mini Exam, Spring A 2025
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This is a closed-book exam: you may not use any books, notes, or electronic devices (calculators, headphones, cell phones, laptops, etc.) during the exam. Mark your answers on the provided blue books. Make sure each answer is clearly indicated. BE SURE TO PUT YOUR NAME ON THE BLUE BOOKS! There are no trick questions on the exam, but you should read the questions carefully.

PLEDGE (required)

The work on this exam will be mine alone, and I will conform with the rules of the exam and the Code of Conduct of the Olin Business School.

Signed name ________________________________

Good luck!

I. Short answer (30 points).

A. In words, what does it mean when one agent is more risk averse than another?

B. What is the difference between priced risk in the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT)?

C. If a client gives you a covariance matrix of returns to use, what should you check? What should you do if there is a problem?

II. Bellman equation (40 points) Consider a continuous-time portfolio choice problem with felicity function $u(c) = \log(c - \bar{c})$ for consumption over an infinite horizon with pure rate of time discount $\rho$. There is a constant riskfree rate $r > 0$ and a single risky asset with constant expected return $\mu > r > 0$ per unit time and constant local variance $\sigma^2 > 0$ per unit time. The choice problem is

Given $w_0 > 0$,
choose adapted $\{\theta_s\}$, $\{c_s\}$, and $\{w_s\}$ to
maximize $\mathrm{E}[\int_{s=0}^{\infty} e^{-\rho s} \log(c_s - \bar{c})]$
s.t. $(\forall s)(dw_s = rw_s ds + \theta_s((\mu - r)ds + \sigma dZ_s) - c_s ds)$
$(\forall s)(w_s \geq 0)$

A. Write down the process $M_t$ for this problem.

B. What does $M_t$ represent given the optimal policies for the portfolio and wealth?

What does $M_t$ represent given an arbitrary candidate policy?

For $t > s$, how can we interpret $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

C. Derive the Bellman equation for this problem.

D. Solve for optimal $c$ and $\theta$ in terms of derivatives of $V$.

E. It is possible to exploit scaling to prove that the value function is of the form $V(w) = \log(w_0 - \bar{c}/r)/\rho + v$ for some constant $v$. (Take it as given.) Use this representation to solve for optimal consumption and portfolio as a function of wealth and $v$.

III. Homotheticity (30 points) Consider an infinite-horizon continuous-time portfolio choice problem with power felicity $u(c) = c^{1-R}/(1-R)$ for consumption, where $R > 0$ and $R \neq 1$, and a pure rate of time discount $\rho > 0$. There is a constant riskfree rate $r > 0$ and a single risky asset with expected return $\mu > r$ per unit time and local variance $\sigma^2 > 0$ per unit time. Then the choice problem is

Given $w_0 > 0$,
choose adapted $\theta_t$, $c_t$, and $w_t$ to
maximize $\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt]$
s.t. $(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt)$
$(\forall t)(w_t \geq 0)$

We want to show that the value function has a special form.

A. Rewrite the choice problem in terms of normalized variables.

B. Note that the optimal solution in the normalized variables is the same independent of $w_0$ (explain why), and use this result to show that the value function can be written in the form $V(w_0) = v{w_0}^{1-R}$.

C. Would the result in part B still be true (perhaps for a different constant $v$) if we added the constraint $(\forall t)c_t \geq 5$? Explain why it is still true or why the proof fails.

IV. Challenger (10 bonus points) Prove the claimed form of the value function in Problem II mentioned in II.E. (This is more difficult than the other questions. I suggest working on this after you have completed and checked the other answers.)

Some formulas that might be useful

univariate Itô's lemma:

Let $dX_t = a_t dt + \sigma_t dZ_t$ where $Z$ is a standard Wiener process, and let $f(X,t)$ have continuous partial derivatives $f_X$, $f_{XX}$, and $f_t$. Then

$$df(X_t,t) = f_X(X_t,t)(a_t dt + \sigma_t dZ_t) + f_t(X_t,t)dt + \frac{\sigma_t^2}{2} f_{XX}(X_t,t)dt.$$

multivariate Itô's lemma:

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where $Z_t$ is an $m$-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY = H_t dt + H_x dX + \frac{1}{2}\mathrm{tr}(GG' H_{xx})dt$$

Note: if $H$ takes values in $\Re^K$, we can apply the result elementwise.

Black-Scholes differential equation:

$$0 = -r\mathcal{O} + \mathcal{O}_t + rS\mathcal{O}_S + \frac{\sigma^2}{2} S^2 \mathcal{O}_{SS},$$

State-price density (stochastic discount factor) if markets are complete:

Let security 0 have a riskless mean return $r$ and any other asset $n = 1, ..., N$ has re-invested risky return $dS_{nt}/S_{nt} = \mu_{nt} t dt + \gamma_{nt} dZ_t$.

$$d\xi = -r dt - (\mu - r\mathbf{1})'(\Gamma')^{-1} dZ_t$$

where

$$\Gamma = (\gamma_1 | \gamma_2 | \ldots | \gamma_N)'.$$

Univariate state-price density:

$$d\xi_t/\xi_t = -r dt - \kappa dZ_t,$$

where $\kappa \equiv (\mu - r)/\sigma$, and with constant coefficients and taking $\xi_0 = 1$ wlog, we have

$$\xi_t = \xi_0 \exp((-r - \kappa^2/2)t - \kappa Z_t),$$

Normal moment generating function:

If $x \sim N(m, s)$, $\mathrm{E}[e^x] = e^{m+s^2/2}$

Arrow-Pratt coefficient of absolute risk aversion:

$\frac{-u''(c)}{u'(c)}$

Arrow-Pratt coefficient of relative risk aversion:

$\frac{-cu''(c)}{u'(c)}$

Constant Absolute Risk Aversion (CARA) utility with risk aversion $A > 0$:

$$u(c) = -\frac{\exp(-Ac)}{A}$$

Constant Relative Risk Aversion (CRRA) utility with risk aversion $R > 0$:

$$u(c) = \begin{cases} \frac{c^{1-R}}{1-R} & \text{for } R \neq 1 \\ \log(c) & \text{for } R = 1 \end{cases}$$

Kuhn-Tucker conditions:

For the optimization model

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$,

the Kuhn-Tucker conditions are

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$